# COT 3100 Fall 2022 Homework #7 Solutions

1) (15 pts) This question considered permutations of "ARTEMISROCKET".

a) How many permutations are there total?

Here are the letter counts: A – 1, R – 2, T – 2, E – 2, M – 1, I – 1 S – 1, O – 1, C – 1, K – 1, 13 total letters. Using the permutation formula, we have $\mathbf{\frac{13!}{2!2!2!} = \frac{13!}{8}}$.

b) How many permutations contain the substring "MISTER"?

Treat "MISTER" as a super letter. Then we have the following other letters: A, R, T, E, O, C and K. Together, these are a total of 8 "unique" letters. The number of ways to arrange these letters is simply **8!.**

c) How many permutations start and end with vowels?

The vowels are as follows: A – 1, E – 2, I – 1, O – 1. The tricky case here is that E appears twice. Let's split up our counting into the following groups: (1) starting and ending letters from AIO, (2) exactly 1 of the starting and ending letters is E, (3) both of the starting and ending letters is E.

(1) There are 3 x 2 = 6 ways to fill in the first and last letters. The frequencies of the remaining letters are 2, 2, 2, 1, 1, 1, 1, and 1. Thus, the last 11 letters in this case can be permuted in $\frac{11!}{2!2!2!}$ ways. It follows that the number of permutations that fall into this case are $\frac{6\times 11!}{2!2!2!}$.

(2) There are 2 x 3 = 6 ways to fill in the first and last letter. To see this, choose the slot for the E (there are two choices), and then the other letter can be one of three things (A, I, O). In this case, the letter frequencies of the remaining letters are 2, 2, 1, 1, 1, 1, 1, 1, and 1. Thus, the last 11 letters in this case can be permuted in $\frac{11!}{2!2!}$ ways. It follows that the number of permutations that fall into this case are $\frac{6\times 11!}{2!2!}$.

(3) We can fill in the E's in only 1 way. The letters that remain have the frequencies 2, 2, 1, 1, 1, 1, 1, 1 and 1. These can be permuted in $\frac{11!}{2!2!}$ ways.

Adding we have: $\frac{6\times 11!}{2!2!2!} + \frac{6\times 11!}{2!2!} + \frac{11!}{2!2!} = \frac{11!}{2!2!}\left(\frac{6}{2} + 6 + 1\right) = \frac{11!}{4} \times 10 = \mathbf{\frac{5(11!)}{2}}$

d) How many permutations do NOT have consecutive vowels in them?

Use the 8 consonants as separators:

___ R __ T __ M __ S __ R __ C __ K __ T __

We have 9 slots to choose from to place the 5 vowels. We choose these slots in $\binom{9}{5}$ ways.

Once the slots are chosen, we permute the vowels in $\frac{5!}{2!}$ ways. Similarly, we can permute the consonants in $\frac{8!}{2!2!}$ ways.

Since each set of choices from each category can be paired with any of the choices from the other categories, it follows that the total number of permutations without consecutive vowels is

$$\binom{9}{5}\frac{5!8!}{2!2!2!} = \binom{9}{5}5!\,7!$$

e) How many permutations contain the vowels in order (A before Es, before I, before O)?

We can select the locations for the vowels in $\binom{13}{5}$ ways. Once the locations are fixed, we can only place the vowels in those locations in 1 way. From here, we must place the 8 consonants (R – 2, T – 2, M – 1, S – 1, C – 1, K – 1) in the 8 remaining spots. We can do this in $\frac{8!}{2!2!}$ ways. It follows that our final answer is the product of these two values: $\binom{13}{5}\frac{8!}{2!2!} = \frac{13!8!}{5!8!2!2!} = \frac{13!}{5!2!2!}$.

Another way of viewing this problem is by noting that there are a total of $\frac{13!}{2!2!2!}$ permutations. But, these permutations include 1 copy for each different ordering of the vowels. (Or, alternately, for a single permutation with the vowels in order there are $\frac{5!}{2!} - 1$, equivalent permutations with the exact same consonants in the same place, but the vowels in a different order. Thus, we can also arrive at our answer by dividing the total number of permutations by $\frac{5!}{2!}$, which yields $\frac{13!}{5!2!2!}$.

2) (15 pts) Consider an ant that is walking on a Cartesian grid, starting at (0,0) and ending at (20, 17). The ant always chooses to walk exactly one unit either up (1 unit positive y direction) or to the right (1 unit positive x direction) whenever he arrives at a Lattice point. (A Lattice point is a point with integer coordinates.) For each of the following restrictions, answer how many ways the ant can make its journey/

a) No restriction other than what is listed above.

We can think of a walking path as 37 unit movements, of which we choose 17 to move +1 in the y direction. We can choose when to do these movements in $\binom{37}{17}$ ways.

b) Ant can not visit the point (13, 6).

First, let's calculate the number of paths that DO visit (13, 6). The number of paths that do are the number of paths to (13, 6) from (0, 0), multiplied by the number of paths from (13, 6) to (20, 17). Using the same logic as before (choosing 6 slots for moving +1 in the y direction), we have $\binom{19}{6}$ ways to get to (13, 6), and we have $\binom{18}{11}$ ways to get from (13, 6) to (20, 17), since we are choosing 11 slots for moving +1 in the y direction for the second portion of the journey.

It follows that the ant can go through (13, 6) in $\binom{19}{6} \times \binom{18}{11}$ ways.

It follows that there are $\binom{37}{17} - \binom{19}{6} \times \binom{18}{11}$ ways the ant can go from (0, 0) to (20, 17) while avoiding (13, 6).

c) Ant can not visit the points (13, 6) and (17, 9).

Using the same strategy as part (b), we'll count the number of paths through (13, 6) and the number of paths through (17, 9) and subtract these both out.

Since we already know the paths through (13, 6) from part (b), let's calculate the number of paths through (17, 9). There are $\binom{26}{9}$ such paths, since we choose 9 out of 26 directions to move in the positive y direction. Then, from (17, 9) to (20, 17) there are $\binom{11}{8}$ paths, since we have 11 more "steps" to take, eight of which will be in the positive y direction. It follows that there are $\binom{26}{9} \times \binom{11}{8}$ paths from (0, 0) to (20, 17) that go through (17, 9).

Subtracting these out as well, we get: $\binom{37}{20} - \binom{19}{13} \times \binom{18}{7} - \binom{26}{9} \times \binom{11}{8}$ paths.

Unfortunately, in this second set, we subtracted out all of the paths that went through both (13, 6) and (17, 9) twice. Thus, via the Inclusion-Exclusion Principle, we must add these back in. In particular, we must multiply the number of paths from (0, 0) to (13, 6) by the number of paths from (13, 6) to (17, 9), by the number of paths from (17, 9) to (20, 17). Two of these have already been calculated, so we just need the number of paths from (13, 6) to (17, 9), which is $\binom{7}{3}$ paths, since we are moving 7 steps, three of which are in the positive y direction. It follows that our final answer is:

$$\binom{37}{17} - \binom{19}{6} \times \binom{18}{11} - \binom{26}{9} \times \binom{11}{8} + \binom{19}{6}\binom{7}{3}\binom{11}{8}$$

d) Ant must visit (2, 7), (5, 9) and (16, 13).

This problem is easier than the previous one. We must need to multiply the product of the number of ways to do four paths:

(1) (0, 0) to (2, 7)
(2) (2, 7) to (5, 9)
(3) (5, 9) to (16, 13)
(4) (16, 13) to (20, 17)

For each, we choose the number of steps we must move in the positive y direction out of the total number of steps to get:

$$\binom{9}{7}\binom{5}{2}\binom{15}{4}\binom{8}{4}$$

e) Ant must visit (13, 10) and (10, 14).

Notice that the ant can't visit both locations using the given constraints. It follows that the answer is 0.

3) (10 pts) A class contains 13 girls and 17 boys. For all parts of this question, each boy and girl are distinguishable from one another. Treat each student as distinguishable from each other student. Answer the following questions:

a) In how many ways can a committee of one boy and one girl be chosen?

Any boy can be paired with any girl, so the answer is 17 x 13 = 221, since each possible committee is a Cartesian product of the boys and girls.

b) In how many ways can a committee of five students be chosen?

There are 30 students total. By definition, we can choose 5 of them in $\binom{30}{5}$ ways.

c) In how many ways can a committee of four girls and three boys be chosen?

Separately count the number of ways 4 girls can be chosen (from the girls) and 3 boys (from the boys). Then, we want the product of these values, since we are essentially taking the Cartesian Product of the set containing sets of 4 girls and the set containing sets of 3 boys. The four girls can be chosen in $\binom{13}{4}$ ways and the three boys can be chosen in $\binom{17}{3}$ ways. It follows that the total nuber of possible committees of four girls and three boys is $\binom{13}{4} \times \binom{17}{3}$.

d) In how many ways can a committee of six students be chosen such that all the students on the committee are the same sex?

We can separately add up the total number of committees of girls with the total number of committees of boys. There are $\binom{13}{6}$ possible committees of girls and $\binom{17}{6}$ possible committees of boys. It follows that the final total is $\binom{13}{6} + \binom{17}{6}$.

e) In how many ways can the girls and boys form a line where no two girls are standing next to one another?

Use the 17 boys as separators. There are 18 gaps between those boys. We can choose 13 of those gaps for the girls in $\binom{18}{13}$ ways. Once the gaps are chosen, we can permute the girls in 13! ways and permute the boys in 17! ways. It follows that the final answer is $\binom{18}{13} 13!\, 17!$. (This can also be written as $\frac{18!17!}{5!}$.)

4) (5 pts) How many integers in between 1 and 42,000,000 are divisible by 6, 15, or 35?

In general, the number of integers divisible by p from the set of integers from 1 to n is $\left\lfloor \frac{n}{p} \right\rfloor$. This is because the last integer out of every p values in every consecutive set of p integers, starting at 1, is divisible by p.

Thus, we can solve this problem by viewing the set of integers as the union of three sets:

Let A be the set of integers in the range divisible by 6.
Let B be the set of integers in the range divisible by 15.
Let C be the set of integers in the range divisible by 35.

Notice that the intersection of A and B is the set of integers in the range divisible by 30 (LCM(6, 15)).
The intersection of A and C is the set of integers in the range divisible by 210 (LCM(6, 35)).
The intersection of B and C is the set of integers in the range divisible by 105 (LCM(15, 35)).
The intersection of A, B and C is the set of integers in the range divisible by 210 (LCM(6, 15, 35)).

If we look at two of the sets above, they are the same set and cancel each other out in the I/E formula. It follows that we can save ourselves a bit of work by skipping both of those calculations. Plugging into the I/E we get:

$$|A \cup B \cup C| = \left\lfloor \frac{42000000}{6} \right\rfloor + \left\lfloor \frac{42000000}{15} \right\rfloor + \left\lfloor \frac{42000000}{35} \right\rfloor - \left\lfloor \frac{42000000}{30} \right\rfloor - \left\lfloor \frac{42000000}{105} \right\rfloor$$

$$= 7{,}000{,}000 + 2800000 + 1200000 - 1400000 - 400000$$

$$= \mathbf{9{,}200{,}000}$$

5) (5 pts) Give a summary of the life and mathematical contributions of Emmy Noether. Please aim for a length of roughly 200 - 400 words. ***Your summary must be typed.*** Please state the sources you used in writing your summary.

Sample Summary

Amalie Emmy Noether was born in 1882 in Erlangen, Germany to a Jewish family. Her father, Max Noether, was a mathematician, leading Emmy to study mathematics at the University of Erlangen, where her father was a lecturer. After completing her dissertation, Noether went to work at the Mathematical Institute of Erlangen for seven years, where she made no income. In 1915, she joined the Universiry of Gottingen, with great resisstence from the faculty due to her being a woman. Mathematician B. L. van der Waerden become acquainted with Noether and use her as a source for a lot of the landmark material that he would go on to publish, such as his 1931 textbook, *Moderne Algebra*. Noether would go on to join the International Congress of Mathematicians in Zurich, becoming one of the most influential mathematicians in the world of algebra. Noether lost her position at the university due to the influence of Germany's Nazi government, after which she went to Bryn Mawr College in Pennsylvania.

During her time at Mathematical Institute of Erlangen, Nother made several contributions to algebraic invariants and number fields. She developed what is now known as Noether's theorem, which has become a staple theorem in theoretical physics and the calculus of variations. At the University of Gottingen, she made further contributions to theoretical physics when she published her paper, *Theory of Ideals in Ring Domains* in 1921, which led to mathematicians describing objects that satisfy an ascending or descending chain conditions as Noetherian. She published several papers on noncommutative algebra and hypercomplex numbers. Noether would also inspire several mathematicians and was credited in numerous papers by other mathematicians that covered a wide range of fields.

Source: https://scientificwomen.net/women/noether-emmy-75