COT 3100 Fall 2022 Homework #6 Solutions

1) (6 pts) Let F_i denote the i[th] Fibonacci number. Prove by induction on n that, for all positive integers n:

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$$

Base case: n = 1, LHS = $\sum_{i=1}^{1} F_{2i} = F_2 = 1$, RHS = $F_{2(1)+1} - 1 = F_3 - 1 = 2 - 1 = 1$
Thus, the assertion holds for n = 1.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{k} F_{2i} = F_{2k+1} - 1$$

Inductive step: Prove for n = k + 1 that

$$\sum_{i=1}^{k+1} F_{2i} = F_{2(k+1)+1} - 1 = F_{2k+3} - 1$$

We prove the assertion by starting with the LHS:

$$\sum_{i=1}^{k+1} F_{2i} = \left(\sum_{i=1}^{k} F_{2i}\right) + F_{2(k+1)}$$

$= F_{2k+1} - 1 + F_{2k+2}$, using the inductive hypothesis
$= F_{2k+3} - 1$, due to the Fibonacci recurrence

This completes the proof of the inductive step. It follows that for all positive integers, n,

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$$

2) (12 pts) Define a sequence, a_i, as follows:

$$a_0 = 0, a_1 = 1, a_2 = 3, a_n = 3a_{n-1} + 2a_{n-2}, for\ all\ ints\ n > 2$$

Using induction on n, prove for all positive integers, n, that

$$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} a_{n+1} & 2a_n \\ a_n & 2a_{n-1} \end{pmatrix}$$

Base case: n = 1 LHS = $\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}^1 = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$, RHS = $\begin{pmatrix} a_{1+1} & 2a_1 \\ a_1 & 2a_{1-1} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

Thus, the assertion holds for n = 1.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} a_{k+1} & 2a_k \\ a_k & 2a_{k-1} \end{pmatrix}$$

Inductive step: Prove for n = k+1 that

$$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} a_{k+2} & 2a_{k+1} \\ a_{k+1} & 2a_k \end{pmatrix}$$

We start proving the inductive step with the LHS:

$$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{k+1} & 2a_k \\ a_k & 2a_{k-1} \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}, \text{ using IH}$$

$$= \begin{pmatrix} 3a_{k+1} + 2a_k & 2a_{k+1} \\ 3a_k + 2a_{k-1} & 2a_k \end{pmatrix}$$

$$= \begin{pmatrix} a_{k+2} & 2a_{k+1} \\ a_{k+1} & 2a_k \end{pmatrix}$$

This proves the inductive step. It follows that for all positive integers, n,

$$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} a_{n+1} & 2a_n \\ a_n & 2a_{n-1} \end{pmatrix}$$

3) (12 pts) Let $a > 1$ be a positive integer. Using induction on n, prove for all positive integers n,

$$(a^2 - a + 1) | ((a-1)^{n+1} + a^{2n-1}).$$

Base case: n = 1, we must show that $(a^2 - a + 1) | \left((a-1)^{1+1} + a^{2(1)-1}\right)$.

$$(a-1)^{1+1} + a^{2(1)-1} = (a-1)^2 + a = a^2 - 2a + 1 + a = (a^2 - a + 1) \times 1$$

Since 1 is an integer, this proves that $(a-1)^{1+1}+a^{2(1)-1}$ is divisible by (a^2-a+1), as desired. Thus, the assertion holds for n = 1.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$(a^2-a+1)|((a-1)^{k+1}+a^{2k-1}).$$

Namely, there exists an integer c such that $((a-1)^{k+1}+a^{2k-1})=c(a^2-a+1)$.

Inductive step: Prove for n = k + 1 that

$$(a^2-a+1)|\left((a-1)^{k+2}+a^{2(k+1)-1}\right).$$

$$\begin{aligned}\left((a-1)^{k+2}+a^{2(k+1)-1}\right) &= (a-1)\times(a-1)^{k+1}+a^{2k+1}\\ &= (a-1)\times(a-1)^{k+1}+a^2a^{2k-1}\\ &= (a-1)\times(a-1)^{k+1}+(a-1)\times a^{2k-1}+(a^2-a+1)\times a^{2k-1}\\ &= (a-1)[(a-1)^{k+1}+a^{2k-1}]+(a^2-a+1)\times a^{2k-1}\\ &= (a-1)c(a^2-a+1)+(a^2-a+1)a^{2k-1}, \text{ using the IH}\\ &= (a^2-a+1)(c(a-1)+a^{2k-1})\end{aligned}$$

Since c is an integer, a is an integer and k is a positive integer,
it follows that $(c(a-1)+a^{2k-1})$ is an integer. This proves the inductive step. It follows that for all positive integers, n,

$$(a^2-a+1)|((a-1)^{n+1}+a^{2n-1})$$

4) (15 pts) Using mathematical induction on n, prove for all positive integers n, that

$$\sum_{i=1}^{n^2} \sqrt{i} \geq \frac{n(4n^2 - 3n + 5)}{6}$$

Hint: In bounding the second sum, you'll have to use two separate lower bounds to replicate the result; one for most of the terms and a different bound for the very last term. (This hint is intentionally cryptic, so don't ask me what it means. If you get to a particular point in the problem, then this hint is more likely to seem relevant.)

Base case: n = 1, LHS = $\sum_{i=1}^{1^2} \sqrt{i} = \sqrt{1} = 1$, RHS = $\frac{1(4(1)^2 - 3(1) + 5)}{6} = \frac{4-3+5}{6} = \frac{6}{6} = 1.$

Thus, LHS ≥ RHS for n = 1 and the assertion is true for n = 1.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{k^2} \sqrt{i} \geq \frac{k(4k^2 - 3k + 5)}{6}$$

Inductive step: Prove for n = k + 1 that

$$\sum_{i=1}^{(k+1)^2} \sqrt{i} \geq \frac{(k+1)(4(k+1)^2 - 3(k+1) + 5)}{6}$$

Let's do some algebra on the RHS so we can state the inductive step more cleanly:

$$\frac{(k+1)(4(k+1)^2 - 3(k+1) + 5)}{6} = \frac{(k+1)(4k^2 + 8k + 4 - 3k - 3 + 5)}{6}$$

$$= \frac{(k+1)(4k^2 + 5k + 6)}{6}$$

$$= \frac{4k^3 + 5k^2 + 6k + 4k^2 + 5k + 6}{6}$$

$$= \frac{4k^3 + 9k^2 + 11k + 6}{6}$$

Thus, the inductive step is to prove the following, for n = k+1

$$\sum_{i=1}^{(k+1)^2} \sqrt{i} \geq \frac{4k^3 + 9k^2 + 11k + 6}{6}$$

$$\sum_{i=1}^{(k+1)^2} \sqrt{i} = \sum_{i=1}^{k^2} \sqrt{i} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{i}$$

$$\geq \frac{k(4k^2-3k+5)}{6} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{i}, \text{ using the IH}$$

$$\geq \frac{k(4k^2-3k+5)}{6} + \left[\sum_{i=k^2+1}^{(k+1)^2-1} k\right] + k + 1$$

The idea here is that all of the terms in the sum (except the last) are greater than k, since the smallest term in the sum is $\sqrt{k^2+1}$, and $\sqrt{k^2+1} > k$, and that the very last term is equal to k+1, so we leave that term the exact same.

$$= \frac{k(4k^2-3k+5)}{6} + k(2k) + k + 1$$

Note that the original second sum had 2k+1 terms, so if we split off 1 term from it, there are 2k terms (each of which is equal to k)

$$= \frac{(4k^3-3k^2+5k)}{6} + \frac{12k^2+6k+6}{6}$$

$$= \frac{4k^3+9k^2+11k+6}{6}$$

This proves the inductive step, as desired. Thus, we can conclude that for all positive integers, n,

$$\sum_{i=1}^{n^2} \sqrt{i} \geq \frac{n(4n^2-3n+5)}{6}$$

5) (5 pts) Give a summary of the academic contributions of Grigori Perelman. Please aim for a length of roughly 200 - 400 words. ***Your summary must be typed.*** Please state the sources you used in writing your summary.

Sample Summary

Grigori Perelman is a Russian mathematician who is best known for proving the Poincare Conjecture. Though many mathematicians shy away from the limelight, Perelman, even amongst mathematicians, is reclusive and prefers not to claim prizes for his work. He was awarded two of the most prestigious prizes in Mathematics: The Clay Mathematics Institute Millennium Prize and the Fields Medal, but refused to claim them.

Perelman was born in Leningrad Russia in 1966. His mother stopped graduate work in mathematics when he was born and recognized his unique talent for mathematics, making sure he was enrolled in the top mathematics programs in Leningrad while growing up. In 1982, Perelman represented the Soviet Union in the International Mathematics Olympiad (IMO), earning a Gold Medal and perfect score. The IMO is a world-wide competition for high school students where each country brings a team of their six best mathematics students. Perelman continued studying mathematics, earning his Ph.D. in 1990, and took a faculty position at the Leningrad Department of Steklov Institute of Mathematics. His top notch work in topology gave him opportunities to do research in the United States. He did research fellowships at both Stony Brook University in New York and UC-Berkeley. In 1994, while in the United States, Perelman proved the soul conjecture for Riemannian geometry. Soon thereafter, even though he received several faculty offers from prestigious American universities, Perelman decided to go back to Russia and work at the Steklov Institute of Mathematics.

After returning to Russia, Perelman started working on a technique called Ricci flow, developed by mathematician Richard Hamilton, in the hopes of proving the Poincare Conjecture. The conjecture states that, "Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere." Hamilton longed to prove the conjecture, but was stuck with a few sticky situations in his proof dealing with singularities. The term "homeomorphic" roughly means, "the same in general structure." For example, if I take a donut shaped stuffed animal and twist it, it might look like a pretzel stick. Even if these two shapes look different, one can be made to look like the other by untwisting it, so the two shapes are homeomorphic. Ultimately, Perelman used a technique called Ricci flow with surgery to deal with the outlying cases that Hamilton had difficulty with, to complete the proof. Perelman published his proof in three papers in 2002 and 2003. It took the mathematical community about three years to verify his proof and in 2006, the proof was accepted and Perelman was awarded the Fields medal. He refused the medal because he didn't believe he deserved it. He felt that Hamilton had done equal or more work to him in terms of proving the Poincare Conjecture. To him, he just cared that the proof was valid and accepted by others in the community, but he didn't want any of the hoopla surrounding it. In fact, it is said that Perelman quit mathematics in 2006 due to this notoriety that he explicitly wanted to avoid. In an article in the New Yorker, Perelman is quoted as saying, "As long as I was not conspicuous, I had a choice. Either to make some ugly thing or, if I didn't do this kind of thing, to be treated as a pet. Now, when I become a very conspicuous person, I cannot stay a pet and say nothing. That is why I had to quit."

To date, the Poincare Conjecture is the only problem of the seven Millennium Prize Problems to have been solved.

Sources
https://en.wikipedia.org/wiki/Grigori_Perelman

https://en.wikipedia.org/wiki/Poincar%C3%A9_conjecture