COT 3100 Fall 2022 Homework #5 Solutions

1) Determine the following summation in terms of n: $\sum_{i=1}^{2^n}(i+8)$. (Note: Please express your answer in the form $2^a + 2^b + 2^c$, where a, b and c are expressions in terms of n.)

$$\sum_{i=1}^{2^n}(i+8) = \sum_{i=1}^{2^n} i + \sum_{i=1}^{2^n} 8$$

$$= \frac{2^n(2^n+1)}{2} + 8(2^n)$$

$$= 2^{n-1}(2^n+1) + 2^3 2^n$$
$$= 2^{2n-1} + 2^{n-1} + 2^{n+3}$$

2) Use an index shift to solve the following summation: $\sum_{i=n}^{2n}(2^{i-n})$.

$$\sum_{i=n}^{2n}(2^{i-n}) = \sum_{i=0}^{n} 2^i = \frac{2^{n+1}-1}{2-1} = \mathbf{2^{n+1}-1}$$

3) Determine the following infinite summation: $\sum_{i=1}^{\infty}(i+5)(\frac{1}{5})^i$.

Let S = $\sum_{i=1}^{\infty}(i+5)(\frac{1}{5})^i$.

Then we have:

$$S = \frac{6}{5} + \frac{7}{5^2} + \frac{8}{5^3} + \cdots$$

Multiply S by $\frac{1}{5}$ and we get:

$$\frac{S}{5} = \frac{6}{5^2} + \frac{7}{5^3} + \cdots$$

Subtract these two lines to get:

$$S - \frac{S}{5} = \frac{6}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$$

$$\frac{4S}{5} = 1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$$

Notice that the RHS is just an infinite geometric sum, so we have:

$$\frac{4S}{5} = \frac{1}{1 - \frac{1}{5}}$$

$$\frac{4S}{5} = \frac{1}{{}^{4}/{}_{5}}$$

$$\frac{4S}{5} = \frac{5}{4}$$

$$S = \mathbf{\frac{25}{16}}$$

More formally, we have $\frac{S}{5} = \sum_{i=1}^{\infty}(i+5)(\frac{1}{5})^{i+1} = \sum_{i=2}^{\infty}(i+4)(\frac{1}{5})^{i}$, by dividing the equation $S = \sum_{i=1}^{\infty}(i+5)(\frac{1}{5})^{i}$, by 5.

Now, subtracting these two equations, we have:

$$S - \frac{S}{5} = \sum_{i=1}^{\infty}(i+5)\left(\frac{1}{5}\right)^{i} - \sum_{i=2}^{\infty}(i+4)(\frac{1}{5})^{i}$$
$$\frac{4S}{5} = \frac{6}{5} + \sum_{i=2}^{\infty}(i+5)\left(\frac{1}{5}\right)^{i} - \sum_{i=2}^{\infty}(i+4)(\frac{1}{5})^{i}$$

$$\frac{4S}{5} = \frac{6}{5} + \sum_{i=2}^{\infty}[(i+5)\left(\frac{1}{5}\right)^{i} - (i+4)(\frac{1}{5})^{i}]$$
$$\frac{4S}{5} = \frac{6}{5} + \sum_{i=2}^{\infty}[((i+5) - (i+4))\left(\frac{1}{5}\right)^{i}]$$
$$\frac{4S}{5} = \frac{6}{5} + \sum_{i=2}^{\infty}[\left(\frac{1}{5}\right)^{i}]$$
$$\frac{4S}{5} = \sum_{i=0}^{\infty}(\frac{1}{5})^{i}$$

$$\frac{4S}{5} = \frac{1}{1 - {}^{1}/{}_{5}}$$

$$\frac{4S}{5} = \frac{5}{4}$$
$$S = \frac{25}{16}$$

4) Let g(n) be defined as follows be a function defined on the positive integers as follows:

g(1) = 2, g(2) = 2, g(3) = 3
For all n > 3, g(n) = 3g(n-1) + g(n-2) + g(n-3).

What are the values of g(4), g(5) and g(6)?

If you would like for fun, write a computer program which prints out the first 1000 values of g(n) mod 10^9+7. Feel free to include the source code inside the document containing your homework solutions. If you want to have LOTS of fun, write a program that reads in a value of n upto 10^{12}, and quickly produces the value of g(n) when divided by $10^9 + 7$. (Hint: For the latter task, you have to embed the recurrence in a matrix and code up fast matrix exponentiation, which is similar to the fast modular exponentiation taught in class.)

g(4) = 3g(3) + g(2) + g(1) = 3(3) + 2 + 2 = **13**
g(5) = 3g(4) + g(3) + g(2) = 3(13) + 3 + 2 = **44**
g(6) = 3g(5) + g(4) + g(3) = 3(44) + 13 + 3 = **148**

Two programs in C are attached as a separate documents.

5) Determine the following matrix product, $\begin{bmatrix} n+1 & n \\ -n+1 & n-1 \end{bmatrix}\begin{bmatrix} 1 & n+1 \\ n-1 & 2 \end{bmatrix}$, in terms of n.

$$\begin{bmatrix} n+1 & n \\ -n+1 & n-1 \end{bmatrix}\begin{bmatrix} 1 & n+1 \\ n-1 & 2 \end{bmatrix} = \begin{bmatrix} (n+1)1 + n(n-1) & (n+1)^2 + 2n \\ (-n+1)1 + (n-1)^2 & (-n+1)(n+1) + 2(n-1) \end{bmatrix}$$

$$= \begin{bmatrix} n+1+n^2-n & n^2+2n+1+2n \\ -n+1+n^2-2n+1 & 1-n^2+2n-2 \end{bmatrix}$$

$$= \begin{bmatrix} n^2+1 & n^2+4n+1 \\ n^2-3n+2 & -(n^2-2n+1) \end{bmatrix}$$

$$= \begin{bmatrix} n^2+1 & n^2+4n+1 \\ (n-1)(n-2) & -(n-1)^2 \end{bmatrix}$$

6) Let n be a positive integer such that 7 | ($5^{2n-1}+2^{2n-1}$). Prove that 7 | | ($5^{2n+1}+2^{2n+1}$). (Hint: Rewrite $5^{2n+1}+2^{2n+1}$, where a portion of the expression is 7 times $5^{2n-1}+2^{2n-1}$, and the rest of the expression is also divisible by 7.)

We are assuming that 7 | ($5^{2n-1} + 2^{2n-1}$). Thus, there exists some integer c such that

$5^{2n-1} + 2^{2n-1} = 7c$.

Now, let’s try to prove the assertion:

$$\begin{aligned}
5^{2n+1} + 2^{2n+1} &= 5^2 5^{2n-1} + 2^2 2^{2n-1} \\
&= 25(5^{2n-1}) + 4(2^{2n-1}) \\
&= 21(5^{2n-1}) + 4(5^{2n-1}) + 4(2^{2n-1}) \\
&= 21(5^{2n-1}) + 4(5^{2n-1}+ 2^{2n-1}) \\
&= 21(5^{2n-1}) + 4(7c), \text{ using given information.} \\
&= 7(3(5^{2n-1})+ 4c)
\end{aligned}$$

It follows that since n is a positive integer and c is an integer that $3(5^{2n-1})+ 4c$ is an integer. Thus we can conclude that 7 | ($5^{2n+1} + 2^{2n+1}$) as desired.

7) Give a summary of life and work of mathematician Maryam Mirzakhani. Please aim for a length of roughly 200 - 400 words. ***Your summary must be typed.*** Please state the sources you used in writing your summary.

Maryam Mirzahkani was a mathematician, who is best known for being the first woman and first Iranian to be honored with a Fields Medal for her work in the geometry of Riemann surfaces and their moduli spaces. She showed an aptitude for mathematics at an early age, earning a Gold Medal in both the Iranian National Olympiad and the International Mathematical Olympiad (IMO) in 1994 and 1995. In the latter year, she earned a perfect score at the IMO. After high school, Mirzahkani attended Sharif University of Technology, followed by graduate school at Harvard University, where she earned her Ph.D. in mathematics in 2004.

After she finished her studies, Dr. Mirzahhani worked at the Clay Mathematics Institute, Princeton University and Stanford University. Her Ph.D. involved solving a previously unsolved problem for coungint the number of simple closed geodescics of length less than L, in terms of L. She showed that this is a polynomial and bounded the polynomial based on the genus of the structure.

She continued her work in this area over the course of the next several years, and was able to prove a conjecture about William Thurston's earthquake flow. In 2014, in collaboration with Alex Eskin and Amir Mohammadi, she proved the regularity of complex geodesics and their closures in modular space. It was for body of this work that she received the Fields Medal in 2014. Unfortunately, Dr. Mirzahkani was diagnosed with breast cancer in 2013and died on July 14, 2017, at the age of 40.

Source: https://en.wikipedia.org/wiki/Maryam Mirzakhani