# Fall 2022 COT 3100 Homework #4 Solutions

1) Find all integer solutions to the equation 1077x + 312y = 21.

Find the GCD of 1077 and 312:
1077 = 3 * 312 + 141
312 = 2 * 141 + 30
141 = 4 * 30 + 21
30 = 1 * 21 + 9
21 = 2 * 9 + **3**
9 = 3 * 3
GCD(1077, 312) = 3

1077 – 3 * 312 = 141
312 – 2 * 141 = 30
141 – 4 * 30 = 21
30 – 1 * 21 = 9
21 – 2 * 9 = 3
21 – 2 * 9 = 3
21 – 2 * (30 – 1 * 21) = 3
21 – 2*30 + 2*21 = 3
3*21 – 2*30 = 3
3(141 – 4*30) – 2*30 = 3
3*141 – 12*30 – 2*30 = 3
3*141 – 14*30 = 3
3*141 – 14(312 – 2*141) = 3
3*141 – 14*312 + 28*141 = 3
31*141 – 14*312 = 3
31(1077 – 3*312) – 14*312 = 3
31*1077 – 93*312 – 14*312 = 3
31 * 1077 – 107 * 312 = 3, now to make RHS = 21, multiply both sides by 7:

(7)((31 * 1077) + (-107 * 312)) = (7)3
217 * 1077 + -749 * 312 = 21

x = 217, y = -749

Thus, we can express all solutions as follows:

$\{(x, y) | (217 - 104c, -749 + 359c), \forall c \in \mathbb{Z}\}$

Note to obtain the offsets 104 and 359, we divided both 312 and 1077 by their gcd, 3.

2) (a) Find all integer solutions to the equation 144x + 83y = 1.

GCD: 1
144 – 1 * 83 = 61
83 – 1 * 61 = 22
61 – 2 * 22 = 17
22 – 1 * 17 = 5
17 – 3 * 5 = 2
5 – 2 * 2 = 1
5 – 2 * 2 = 1

5 – 2 * (17 – 3 * 5) = 1
5 – 2*17 + 6*5 = 1
7*5 – 2*17 = 1
7(22 – 17) – 2*17 = 1
7*22 – 7*17 – 2*17 = 1
7*22 – 9*17 = 1
7*22 – 9(61 – 2*22) = 1
7*22 – 9*61 + 18*22 = 1
25*22 – 9*61 = 1
25(83 – 61) – 9*61 = 1
25*83 – 25*61 – 9*61 = 1
25*83 – 34*61 = 1
25*83 – 34(144 – 83) = 1
25*83 – 34*144 + 34*83 = 1
59*83 – 34*144 = 1
Thus, one solution is: x = -34, y = 59. All solutions are described in the set below:

$\{(x, y) | (-34 + 83c, 59 - 144c), \forall c \in \mathbb{Z}\}$

(b) Find all integer solutions to the equation 144x + 83y = 11.

From our previous solution:
(59 * 83 – 34 * 144) = 1, now multiply this through by 11:

(11)(-34 * 144 + 59 * 83) = 1 * 11
144(-374) + 83(649) = 11
Thus, x = -374, y = 649 is one solution and the whole set of solutions is:

$\{(x, y) | (-374 + 83c, 649 - 144c), \forall c \in \mathbb{Z}\}$

We can also “re-center” the solution set and describe it as follows:

$\{(x, y) | (-42 + 83c, 73 - 144c), \forall c \in \mathbb{Z}\}$

(c) Find $83^{-1}$ mod 144. (Note: Answer must be in between 0 and 143, inclusive.)

Take the last line of the Extended Euclidean Algorithm from part (a) and evaluate it mod 144:

59 * 83 – 34 * 144 = 1

$59 * 83 - 34 * 144 \equiv 1 \ (mod\ 144)$
$59 * 83 \equiv 1 \ (mod\ 144)$ , since $144 \equiv 0 \ (mod\ 144)$

It follows that $83^{-1} \equiv \mathbf{59}\ (\boldsymbol{mod}\ \mathbf{144})$

3) Let a = $2^2 3^9 5^8 7^4$, b = $2^7 3^6 5^8 11^4$, and c = $2^5 3^5 5^{10} 11^3$. Determine, in prime factorized form, both gcd(a, b, c) and lcm(a, b, c).

gcd(a, b, c) = $2^2 3^5 5^8$
lcm(a, b, c) = $2^7 3^9 5^{10} 7^4 11^4$

4) For the numbers a, b and c listed in problem 4, determine the number of divisors each of those numbers has.

a) $2^2 3^9 5^8 7^4$
$\Pi(q_i + 1)$ = 3*10*9*5 = 1350 divisors.

b) $2^7 3^6 5^8 11^4$
$\Pi(q_i + 1)$ = 8*7*9*5 = 2520 divisors.

c) $2^5 3^5 5^{10} 11^3$
$\Pi(q_i + 1)$ = 6*6*11*4 = 1584 divisors.

5) How many zeroes are at the end of $\frac{3000!}{1500!1500!}$?

First calculate how many zeroes are at the end of 3000!:

$\frac{3000}{5} = \frac{600}{5} = \frac{120}{5} = \frac{24}{5} = \frac{4}{5} = 0$, so 600 + 120 + 24 + 4 = 748 zeros in the numerator.

Next, calculate the number of zeroes at the end of 1500!:

$\frac{1500}{5} = \frac{300}{5} = \frac{60}{5} = \frac{12}{5} = \frac{2}{5} = 0$, so 300 + 60 + 12 + 2 = 374 zeros.

This means that the numerator equals $x10^{748}$, where x is an integer that doesn't end in 0 and the denominator equals $(y10^{374})^2 = y^2 10^{748}$, where y is an integer that doesn't end in 0. It follows that result of the division is an integer with **0 trailing zeroes.**

6) Prove that $\gcd(a,b) \times \gcd(a,c) \times \gcd(b,c) \geq (\gcd(a,b,c))^3$ for all positive integers, a, b, and c. Intuitively, without proof (though if you can provide a proof that would be great), determine the situations where the two sides of this equation are equal. (Note: The originally posed question is relatively easy, so be looking for straight-forward observations about the property of the gcd function as opposed to something detailed about esoteric.)

Let d = gcd(a,b,c).

The greatest common divisor of three values must be equal to or smaller than the greatest common divisor of two of those values.

Another key observation is that gcd(a,b) is always a multiple of gcd(a,b,c). We can ascertain this by looking at the prime factorization GCD formula, or noting that by definition, gcd(a, b, c) | gcd(a, b), since anything that divides evenly into a, b and c, must divide evenly into a and b.

Thus, there exist integers x, y and z such that gcd(a, b) = dx, gcd(a, c) = dy and gcd(b, c) = dz.

Therefore, we have

$$\gcd(a,b) \times \gcd(a,c) \times \gcd(b,c) = dx \times dy \times dz = d^3(xyz) \geq d^3 = (\gcd(a,b,c))^3$$

Since the greatest common divisor of three integers a, b, and c, must be an integer itself, x, y, and z are positive integers. Thus, by the definition of multiplication, $d^3xyz$ must be greater than or equal to $d^3$.

Therefore $\gcd(a,b) \times \gcd(a,c) \times \gcd(b,c) \geq (\gcd(a,b,c))^3$.

These two quantities can ONLY be equal if x = y = z = 1. Let's examine when this occurs. If gcd(a, b) = gcd(a, b, c), then there's no prime divisor that a and b share, that c doesn't share. Similarly, there's no prime divisor that a and c share that b doesn't share and finally there's no prime divisor that b and c share that a doesn't share.

If we let the exponents of a, b and c in their prime factorizations be $a_i$, $b_i$ and $c_i$, for prime $p_i$, then $\min(a_i, b_i) = \min(a_i, c_i) = \min(b_i, c_i) = \min(a_i, b_i, c_i)$. This can not be true if the values $a_i$, $b_i$ and $c_i$ are all distinct. In fact, it ONLY holds if the second smallest value of the three equals the smallest value. Thus, for each prime $p_i$, in the prime factorizations of a, b and c, we must have that = $\min(a_i, b_i, c_i) = \text{mid}(a_i, b_i, c_i)$, where mid(x, y, z) represents the middle value of x, y and z.

7) Give a summary of the life and mathematical contributions of Carl Friedrich Gauss.

Carl Friedrick Gauss was a German mathematician from the 18th and 19th centuries. He has been referred to as the Princeps mathematicorum, or commonly as one of the greatest mathematicians of all time. In his impressive life he contributed to algebra the proof for things like the fundamental theorem of algebra, Descartes' rule of signs, introduced the ≡ symbol. This was all in addition to his extensive achievements in astronomy, topology, and magnetism.

Gauss was born in Brunswick, in what is now Lower Saxony, Germany. His mother was illiterate and only knew faint details about his birthdate, but later in his life Gauss was able to use that information to calculate his actual date of birth based on the date of Easter. Throughout his early life. the man was a prodigy. Anecdotes have been floated about his inhuman mathematical feats as early as the age of three, but what is known for sure is that he completed one of the most influential documents on number theory, the Disquisitiones Arithmeticae, at only 21 years old. Gauss went on to attend multiple universities at which he rediscovered multiple important theorems of geometry and algebra, and because of which decided to choose mathematics as his career.

Even while suffering through gout and depressive moods, Gauss was publishing impressive works. His Dioptrische Untersuchungen, a feat of geometric optics and a pioneering work in raytracing, was published when he was 63. Throughout his late life into his 70s Gauss received accolade after accolade, becoming a member of the Royal Institute of the Netherlands and the American Philosophical Society. He sadly ended up dying of a heart attack in 1855, however.

(https://en.wikipedia.org/wiki/Carl_Friedrich_Gauss)