Fall 2022 COT 3100 Homework #2 Solutions

1) (4 pts) Prove the following set equality properties by membership table:

$$(A \cap C) \cup (C - B) = C - (B - A).$$

Solution

A	B	C	$A \cap C$	$C - B$	$(A \cap C) \cup (C - B)$	$B - A$	$C - (B - A)$
0	0	0	0	0	0	0	0
0	0	1	0	1	1	0	1
0	1	0	0	0	0	1	0
0	1	1	0	0	0	1	0
1	0	0	0	0	0	0	0
1	0	1	1	1	1	0	1
1	1	0	0	0	0	0	0
1	1	1	1	0	1	0	1

2) (4 pts) Prove the following set equality properties by membership table:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Solution

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$(A \cap B)$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

3) (4 pts) Prove by simplification using set equality laws that

$$(\bar{A} \cap ((B \cap C) \cup (B \cap \bar{C}))) \cup \bar{A} = \bar{A}.$$

Solution

$$\left(\bar{A} \cap \left((B \cap C) \cup (B \cap \bar{C})\right)\right) \cup \bar{A} = \bar{A} \cup \left(\bar{A} \cap \left((B \cap C) \cup (B \cap \bar{C})\right)\right) \qquad \text{Commutative Law}$$

$$= \bar{A} \qquad \text{Absorption Law}$$

Note: I didn't necessarily mean for this question to be so easy…when I made it, I built it backwards with several steps, and at the time, didn't recognize that you could just apply Commutative followed by Absorption to show the equality.

4) (4 pts) Prove by simplification using set equality laws that

$$A \cup \left(\overline{A \cap (\overline{\bar{B} \cap \bar{A}})}\right) = \mathcal{U}$$

Solution

$$A \cup \left(\overline{A \cap (\overline{\bar{B} \cap \bar{A}})}\right) = A \cup (\bar{A} \cup \overline{\overline{\bar{B} \cap \bar{A}}}) \qquad \text{De Morgan's Law}$$

$$= A \cup (\bar{A} \cup (\bar{B} \cap \bar{A})) \qquad \text{Double Negation}$$

$$= A \cup (\bar{A} \cup (\bar{A} \cap \bar{B})) \qquad \text{Commutative Law}$$

$$= A \cup \bar{A} \qquad \text{Absorption Law}$$

$$= \mathcal{U} \qquad \text{Inverse Law}$$

Please follow these directions for questions 5 - 8:

Prove the following subset and set equality properties by considering arbitrary elements of the subset and applying the principle of universal generalization. (For set equalities, **<u>you have to do this twice</u>**.)

5) (3 pts) $(A \cup B) \cap C \subseteq (A \cap B) \cup C$

Solution

Let x be an arbitrarily chosen element in the set $(A \cup B) \cap C$.
Our goal is to show that $x \in (A \cap B) \cup C$.

Be definition of intersection, we have $x \in (A \cup B)$, and $x \in C$.

Since $p \to (p \vee q)$, it follows that $x \in C \ \vee x \in (A \cap B)$, thus, by the commutative law we have $x \in (A \cap B) \vee x \in C$. Finally, by the definition of union, we have $x \in (A \cap B) \cup C$, as desired.

6) (4 pts) $A - B - C \subseteq A - (B - C)$

Solution

Let $x \in A - B - C$, arbitrarily chosen. We desire to prove that $x \in A - (B - C)$.

By definition of set difference, we have $x \in A \wedge x \notin B \wedge x \notin C$.
In order for an element to be in B – C, it must be an element of B.
But, since $x \notin B$, it follows that $x \notin B - C$.
Since $x \in A \wedge x \notin B - C$, by definition of set difference, it follows that $x \in A - (B - C)$, as desired.

7) (6 pts) $(A \cup \bar{B} \cup \bar{C}) = \mathcal{U} - ((B \cap C) - A)$

Solution

To show set equality, we must show that each set is a subset of the other.

First, let's prove$(A \cup \bar{B} \cup \bar{C}) \subseteq \mathcal{U} - ((B \cap C) - A)$, using direct proof.
Let $x \in (A \cup \bar{B} \cup \bar{C})$. Thus, $x \in A \vee x \notin B \vee x \notin C$, by definition of union.

Note that if $x \in A$, then by definition of set difference, $x \notin B \cap C - A$.

Now we consider the other possibility: $x \notin B \vee x \notin C$. By definition of set intersection, if x is not a member of either B or C, then it follows that $x \notin B \cap C$, since it can not be in both sets B and C. By definition fo set difference, we conclude that in this case $x \notin B \cap C - A$.

Thus, no matter which option of the three initial options is true about x, it follows that $x \notin B \cap C - A$, Thus, by definition of the universe and set difference, we have $x \in \mathcal{U} - ((B \cap C) - A)$, as desired.

Now we prove that $\mathcal{U} - \big((B \cap C) - A\big) \subseteq (A \cup \bar{B} \cup \bar{C})$.
Let $x \in \mathcal{U} - \big((B \cap C) - A\big)$.
By definition of set difference $x \notin (B \cap C) - A$.
It follows by definition of set difference (again) that either $x \notin (B \cap C)$ or $x \in A$. (Technically, we are applying De Morgan's law to the negation of an and.)

We have two cases: (a) $x \notin (B \cap C)$ or (b) $x \in A$.

In case (a), by definition of intersection, we have either $x \notin B \vee x \notin C$. By definition of set complement this means either that $x \in \bar{B} \vee x \in \bar{C}$. By definition of or and set union, it follows that $x \in A \vee x \in \bar{B} \vee x \in \bar{C}$.

In case (b), by definition of union, it follows that that $x \in A \vee x \in \bar{B} \vee x \in \bar{C}$.

Thus, in both cases, since $\in A \vee x \in \bar{B} \vee x \in \bar{C}$, we can conclude by definition of set union that $x \in (A \cup \bar{B} \cup \bar{C})$, as desired.

8) (6 pts) $A \cap (A \cup B) = A$

Solution
We must show (1) $A \cap (A \cup B) \subseteq A$ and (2) $A \subseteq A \cap (A \cup B)$.

Let's prove (1) via direct proof.
Let $x \in A \cap (A \cup B)$. We aim to prove that $x \in A$.
By definition of intersection, $x \in A \wedge x \in (A \cup B)$.
By conjunctive simplification, we have $x \in A$, as desired.

Now, let's prove (2) via direct proof.
Let $x \in A$. We aim to prove that $x \in A \cap (A \cup B)$.
By definition of or and union, given that $x \in A$, it follows that $x \in A \cup B$.
Since we've derived that $x \in A$ and $x \in A \cup B$, by definition of intersection it follows that $x \in A \cap (A \cup B)$, as desired.

9) (5 pts) Prove or disprove for arbitrary sets *A*, *B* and *C*:

$$\text{if } B \subseteq C \text{, then } B - A \subseteq C - A.$$

Solution
The assertion is true.

We will prove that $B - A \subseteq C - A$ under the assumption that $B \subseteq C$.
To prove $B - A \subseteq C - A$, we start with an arbitrarily chosen element $x \in B - A$.
We aim to prove that $x \in C - A$.

By definition of set difference, we have $x \in B \wedge x \notin A$.
Since $x \in B \wedge B \subseteq C$, by definition of subset, it follows that $x \in C$.
Since $x \in C \wedge x \notin A$, by definition of set difference, it follows that $x \in C - A$, as desired.

10) (5 pts) Prove or disprove for arbitrary sets *A*, *B* and *C*:

$$\text{if } A \subseteq B \text{ and } C \subseteq D \text{, then } A \cup C \subseteq B \cap D$$

Solution
This assertion is false. Consider the following counter-example:

A = { 1 }
B = { 1 }
C = { 2 }
D = { 2 }

In this example, $A \subseteq B$ and $C \subseteq D$, since both pairs of sets are equal. But, $A \cup C = \{1,2\}$ while $B \cap D = \emptyset$. Thus, $A \cup C \nsubseteq B \cap D$, disproving the assertion.

11) (5 pts) Give a summary of the life (on going) and mathematical contributions of Ingrid Daubechies.

Sample Summary

Dr. Daubechies is currently the James B. Duke professor for Duke University's Electrical and Computer Engineering program. She had a fascination with numbers form an extremely young age in her native Belgium, where she did her schooling, culminating in earning her Ph. D. in theoretical physics in 1980 from the Free University in Brussels.

She started off her career with a professorship at Vrije Universiteit Brussel in the early 1980s, moving to a researcher position at the Courant Institute of Mathematical Sciences in 1986, where she discovered that wavelets could be utilized to aid digital signal processing. It's from this work that she developed what is now known as the Daubechies wavelet and the biorthogonal CDF wavelet. The JPEG 2000 standard uses a wavelet from this group, and this is what she's best known for. In addition to wavelets, she has used mathematics and technology to automatically extract information from bones and teeth and she's created advanced image processing techniques to verify the authenticity of famous paintings.

Over the course of her academic career, Daubechies has taught at Rutgers, Princeton and now Duke. She has routinely helped other women in mathematics and science. She is on the board of the Enhancing Diversity in Graduate Education (EDGE) program that helps women applying to graduate school and she started a summer program at Duke in mathematics for girls who are rising high school seniors. She has won many awards over her illustrious career, including the North American Laureate in 2019. The award has been given annually since 1988 to five women in the fields of chemistry, physics, materials science, mathematics and computer science.

Source: https://en.wikipedia.org/wiki/Ingrid_Daubechies