

COT 3100 Fall 2022 Homework #10 Solutions

1) (14 pts) Let R_1 and R_2 be relations on a set $A = \{1, 2, 3, 4\}$.
In particular, let $R_1 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 2)\}$ and
 $R_2 = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 3), (3, 4), (4, 4)\}$.

Determine the following:

- Whether or not R_1 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
- Whether or not R_2 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
- The relation $R_1 \circ R_2$.
- The relation $R_2 \circ R_1$.
- $R_1 \cup R_2$
- $R_1 \cap R_2$
- The reflexive, symmetric and transitive closures of both R_1 and R_2 .

Solution

- R_1 is not reflexive because it does not contain $(1, 1)$.
 R_1 is not irreflexive because it contains $(2, 2)$.
 R_1 is symmetric because for all ordered pairs of the form (a,b) it also contains (b,a) .
Specifically, for $a \neq b$, the only ordered pairs are $(1, 3)$, $(3, 1)$, $(2, 4)$ and $(4, 2)$ which “match.”
 R_1 is not anti-symmetric because it contains both $(1, 3)$ and $(3, 1)$.
 R_1 is not transitive because it contains both $(1, 3)$ and $(3, 1)$, but doesn't contain $(1, 1)$.
- R_2 is reflexive because it does contains $(1, 1)$, $(2, 2)$, $(3, 3)$ and $(4, 4)$.
 R_2 is not irreflexive because it contains $(1, 1)$.
 R_2 is not symmetric because it contains $(1, 3)$ but doesn't contain $(3, 1)$.
 R_2 is anti-symmetric. The only ordered pairs of the form (a, b) and (b, a) that are in R_2 are $(1, 1)$ and $(1, 1)$, $(2, 2)$ and $(2, 2)$, $(3, 3)$ and $(3, 3)$, and $(4, 4)$ and $(4, 4)$. In each case, $a = b$.
 R_2 is not transitive because it contains both $(2, 3)$ and $(3, 4)$ but doesn't contain $(2, 4)$.
- $R_1 \circ R_2 = \{(1, 1), (1, 3), (2, 1), (2, 2), (2, 4), (3, 1), (3, 2), (4, 2)\}$
- $R_2 \circ R_1 = \{(1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 1), (3, 3), (4, 2), (4, 3)\}$
- $R_1 \cup R_2 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 2), (1, 1), (2, 3), (3, 3), (3, 4), (4, 4)\}$
- $R_1 \cap R_2 = \{(1, 3), (2, 2)\}$
- $r(R_1) = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 2), (1, 1), (3, 3), (4, 4)\}$
 $s(R_1) = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 2)\}$, it's already symmetric.
 $t(R_1) = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 2), (1, 1), (3, 3), (4, 4)\}$
 $r(R_2) = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 3), (3, 4), (4, 4)\}$, it's already reflexive.
 $s(R_2) = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 3), (3, 4), (4, 4), (3, 1), (3, 2), (4, 3)\}$
 $t(R_2) = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 3), (3, 4), (4, 4), (1, 4), (2, 4)\}$

2) (5 pts) Let R be a relation over the positive integers defined as follows:

$$R = \{(a,b) \mid a, 2a \text{ and } b \text{ form side lengths of a triangle with positive area} \}$$

Determine whether or not R satisfies the following properties. Give a brief justification for each of your answers.

- (i) reflexive
- (ii) irreflexive
- (iii) symmetric
- (iv) anti-symmetric
- (v) transitive

Solution

R is NOT reflexive because $(1, 1)$ is not in R . $(1, 1)$ is not in R because a triangle with side lengths 1, 2 and 1 is degenerate (no area) because $1 + 1 = 2$.

R IS irreflexive. For all positive integers a , (a, a) is NOT in R because a triangle can NOT have side lengths a , $2a$ and a because $a + a = 2a$ (and is not greater than the third side length).

R is not symmetric. $(2, 3)$ is in R but $(3, 2)$ is not. To see this, note that a triangle does exist with side lengths 2, 4 and 3 (since $2 + 3 > 4$), but no triangle exists with side lengths 3, 6 and 2 since $(2 + 3 < 6)$.

R IS anti-symmetric. We can prove for all integers $b \leq a$, that a , $2a$ and b are NOT side lengths of a triangle with positive area because $a + b \leq a + a = 2a$, so two of the side lengths sum up to no more than the third side length. If this is the case, then if (a, b) is in the relation, we know that $b > a$, and this means that (b, a) can NOT be in the relation because b , $2b$ and a can NOT form side lengths of a triangle when $b > a$.

R is NOT transitive. $(2, 4)$ is in R and $(4, 8)$ is in R , but $(2, 8)$ is not. To see this, note that 2, 4 and 4 form an isosceles triangle as do 4, 8 and 8. But, 2, 4 and 8 do not form a triangle because $2 + 4 < 8$.

3) (6 pts) How many anti-symmetric relations on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ contain the ordered pairs $(1, 1)$, $(3, 3)$, $(4, 4)$, $(5, 8)$, and $(9, 1)$?

Solution

For a relation to be anti-symmetric, for all ordered pairs of the form (a, b) and (b, a) with $a \neq b$, one of three options must be true: (a) none are in the set, (b) (a, b) is in the set only, or (c) (b, a) is in the set only.

Thus, to count how many anti-symmetric relations there could be on the set A , partition the set into 55 sets, 45 sets of the form $\{(a, b), (b, a)\}$ where $a \neq b$, and 10 sets of the form $\{(a, a)\}$.

If we are given no information about a set of the form $\{(a, b), (b, a)\}$, then there are 3 options for valid order pairs we can put in R .

If we are given no information about a set of the form $\{(a, a)\}$, then there are 2 options (include the item or don't include it) for valid ordered pairs we can put in R.

Out of the 10 sets of the form $\{(a,a)\}$ we are told that for 3 of them, we must include the ordered pair $(a = 1, 3, 4)$. For the other 7 values, we have 2 choices each. Thus, there are 2^7 ways in which we can assign the membership of the ordered pairs $(2, 2), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)$, and $(10, 10)$.

For the ordered pairs $(5, 8)$ and $(9, 1)$ we are forced to include them AND forced not to include $(8, 5)$ and $(1, 9)$. Thus, we only have 1 option with respect to these sets: $\{(5, 8), (8, 5)\}$ and $\{(9, 1), (1, 9)\}$. For the other 43 sets of the form $\{(a,b), (b,a)\}$ with $a \neq b$, we have 3 options each of what to include in R. It follows that we can do this in 3^{43} ways.

Since each selection for the sets of the form $\{(a,a)\}$ can be paired with each selection for the sets of the form $\{(a,b),(b,a)\}$ with $a \neq b$, it follows that there are $2^7 3^{43}$ possible anti-symmetric relations over the set A which contain the given ordered pairs.

4) (4 pts) Let $P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$. $P(4) = P(5) = P(6) = P(7) = P(8) = 0$. What is the value of $a - b + c - d + e$?

Solution

Since the given equation is a quintic and we are given 5 unique roots of the equation, we can re-express $P(x) = (x - 4)(x - 5)(x - 6)(x - 7)(x - 8)$.

Finally, note that $P(-1) = -1 + a - b + c - d + e$, by plugging x into the polynomial form.

$$\text{But, } P(-1) = (-1 - 4)(-1 - 5)(-1 - 6)(-1 - 7)(-1 - 8) = -\frac{9!}{4!} = -15120$$

$$\text{Thus, } -15120 = -1 + a - b + c - d + e$$

$$\mathbf{-15119 = a - b + c - d + e}$$

5) (4 pts) Let $f(x) = x^2 + 8x - 9$ with a domain of all real $x \in [-\infty, -4]$. Prove that f is injective. What is the range of f ? (You may either use Calculus or complete the square to prove your answers.)

Solution

To prove that f is injective, we must show that if $f(x) = f(y)$, then $x = y$. Note that $x, y \leq -4$.

Let x and y be values such that $f(x) = f(y)$, so

$$x^2 + 8x - 9 = y^2 + 8y - 9$$

$$x^2 - y^2 + 8x - 8y = 0$$

$$(x - y)(x + y) + 8(x - y) = 0$$

$$(x - y)(x + y + 8) = 0$$

Either $x - y = 0$, which means $x = y$, as desired, OR

$x + y + 8 = 0$, but if $x, y \leq -4$, the only solution is $x = y = -4$, and we still reach the desired conclusion that $x = y$.

Thus, we've proven the given function is injective.

The range of f is $[-25, \infty]$.

6) (4 pts) Find $f^{-1}(x)$ for the function given in question #5.

Solution

Swap x and y and then solve for y :

$$y^2 + 8y + 16 - 9 = x + 16$$

$$y^2 + 8y + 16 = x + 25$$

$$(y + 4)^2 = x + 25$$

$$y + 4 = -\sqrt{x + 25}, \text{ this is because } y \leq -4$$

$$y = -4 - \sqrt{x + 25}$$

$$\text{Thus, } f^{-1}(x) = -4 - \sqrt{x + 25}.$$

7) (8 pts) Let $f(x) = ax^3 + bx^2 + cx + d$ and $f(-1) = -6$, $f(0) = 4$ and $f(1) = 12$.

(1) (1 pt) What is the value of d ?

(2) (3 pts) What is the value of b ?

(3) (2 pts) What is the value of $a+c$?

(4) (2 pts) Prove that the value of a is not uniquely determined by finding two sets of ordered quadruplets (a_1, b_1, c_1, d_1) and (a_2, b_2, c_2, d_2) with $a_1 \neq a_2$ that is consistent with all of the given information above.

Solution

$$f(0) = d = 4, \text{ so } \mathbf{d = 4}.$$

$$f(1) = a + b + c + 4 = 12$$

$$f(-1) = -a + b - c + 4 = -6$$

$$\text{-----}$$
$$f(1)+f(-1) = 2b + 8 = 6$$

$$2b = -2$$

$$\mathbf{b = -1}$$

Plug in $b = -1$ for the equation for $f(1)$ to get $a + (-1) + c + 4 = 12$

$$\mathbf{a + c = 9}$$

First, let's try $a = 9$, $b = -1$, $c = 0$ and $d = 4$ for the polynomial $f(x) = 9x^3 - x^2 + 4$. Notice that indeed $f(-1) = -6$, $f(0) = 4$ and $f(1) = 12$.

Next, let's try $a = 1$, $b = -1$, $c = 8$ and $d = 4$ for the polynomial $f(x) = x^3 - x^2 + 8x + 4$. Again, $f(-1) = -6$, $f(0) = 4$ and $f(1) = 12$.

In fact, any two polynomials with $a + c = 9$, $b = -1$ and $d = 4$ will suffice.

8) (5 pts) Please give a summary of the life and mathematical contributions of Srinivasa Ramanujan. Please aim for a length of roughly 200 - 400 words. **Your summary must be typed.** Please state the sources you used in writing your summary. If you are so inclined, after finals, go watch the movie "The Man Who Knew Infinity."

Sample Write Up (based on the Wikipedia entry for Ramanujan)

Srinivasa Ramanujan is to mathematics what many consider Mozart to be for music. With minimal formal training in mathematics, Ramanujan derived some of the most complicated known mathematics and even new theorems, completely on his own.

Born in a small village outside of Madras, India in 1887, Ramanujan very quickly showed his affinity for mathematics. By age 11, soon after he had been introduced to formal mathematics in school, he was able to exhaust all of the knowledge of two college boarders who lived at his house. At the age of 16, he obtained a copy of a pure mathematics book and devoured its contents. Using this book as a starting point, he calculated the Euler-Mascheroni constant to 15 decimal places. Soon after, Ramanujan attempted to attend college, but due to the fact that all he cared about was mathematics, he earned poor grades in other subjects. After completing school, he spent most of free time doing mathematics, but ultimately had to get a job. He landed a job as a clerk in a revenue department. Upon seeing Ramanujan's notebooks, his boss realized that he didn't deserve such a boring job.

Ramanujan's boss showed the notebooks to other mathematicians in the area. Eventually, these notebooks circulated among a few British mathematicians, several of whom felt that though Ramanujan may have some talent, he lacked formal training and rigor in his proofs. Finally, however, two famous British mathematicians, G. H. Hardy and J. E. Littlewood, felt that Ramanujan's genius deserved to be discovered by others. They offered to have Ramanujan come to England to do mathematical research with them. In 1914, at the age of 27, Ramanujan took a ship from India to England to join Hardy and Littlewood. In working with Hardy and Littlewood, they improved the rigor of the proofs of Ramanujan's results, as often times, Ramanujan relied on his instincts. In 1918, he was elected a Fellow of the Royal Society, one of the youngest mathematicians to do so and only the second Indian ever, at the time.

Throughout his life, Ramanujan experienced health problems. Though it was typically thought that he died of tuberculosis, some now believe that he actually died from amoebiasis, a treatable disease that was widespread in Madras at the turn of the 20th century. Ramanujan passed away on April 26, 1920. In six short years of work with Hardy and Littlewood, he produced a great quantity of results. Some of the well-known results he proved dealt with the problem of partitions, the number of ways to divide a positive integer into a set of positive integers. (For example, 4 can be partitioned as follows: 4, 2+2, 1+3, 1+1+2, and 1+1+1+1.), the Bernoulli numbers and coming up with series that can be used to generate many digits of π quickly.