

Fall 2022 COT 3100 Homework #1 Solutions

1) (6 pts) Joanna and Ahmed are running a 13 mile race. For the first three miles, both run at the same speed. Unfortunately, Ahmed got a little tired at that point, so for the last ten miles, he ran 2 miles per hour slower than his original speed while Joanna continued to run at the same speed she ran the first three miles. If Ahmed finished the race exactly 25 minutes after Joanna did.

- (a) At what speed (in miles per hour), did both start running at the beginning of the race?
- (b) How long did Joanna take to finish the race (answer in hours:minutes:seconds)?
- (c) How long did Ahmed take to finish the race (answer in hours:minutes:seconds)?

Solution

Let r be the speed they run at, in miles per hour, at the beginning of the race.

Let t represent the time it took Joanna to run the last 10 miles of the race.

For the last 10 miles of the race, Ahmed runs at $r-2$ mph while Joanna still runs at r mph.

Ahmed takes 25 minutes = $\frac{5}{12}$ more hours than Joanna does to run the same distance. So we have:

$$\begin{aligned}10 &= rt \\10 &= (r - 2)\left(t + \frac{5}{12}\right)\end{aligned}$$

Use the first portion of the equation to solve for r : $r = \frac{10}{t}$, and substitute this into the second equation:

$$\begin{aligned}10 &= \left(\frac{10}{t} - 2\right)\left(t + \frac{5}{12}\right) \\10 &= 10 + \frac{25}{6t} - 2t - \frac{5}{6} \\0 &= -2t - \frac{5}{6} + \frac{25}{6t}\end{aligned}$$

This equation will be easier to handle if we multiply it through by $6t$ and put everything to the other side:

$$\begin{aligned}12t^2 + 5t - 25 &= 0 \\(3t + 5)(4t - 5) &= 0\end{aligned}$$

Since t must be positive, it follows that $(4t - 5) = 0$, and $t = \frac{5}{4} \text{ hrs} = 1 \text{ hr } 15 \text{ min}$.

(a) Now, we can solve for r : $r = \frac{10}{t} = \frac{10 \text{ miles}}{\frac{5}{4} \text{ hr}} = \mathbf{8 \text{ mph}}$.

(b) Joanna took $\frac{13 \text{ miles}}{8 \text{ mph}} = 1\frac{5}{8} \text{ hr} = 1 \text{ hr } \frac{75}{2} \text{ min} = \mathbf{1 \text{ hr, } 37 \text{ minutes, } 30 \text{ seconds}}$ (Typically written as 1:37:30, to finish the race.

(c) Ahmed finished the race 25 minutes later so he took **2 hours, 2 minutes and 30 seconds**.

2) (6 pts) Determine the ordered pair, (x, y), which satisfies the following pair of equations:

$$\log_2(x^2) + \log_4 y = 7$$

$$\log_4(8x) + \log_2(2y^2) = 8$$

Solution

Changing bases, we have:

$$2\log_2 x + \frac{\log_2 y}{\log_2 4} = 7$$
$$\frac{\log_2(8x)}{\log_2 4} + \log_2 2 + 2\log_2 y = 8$$

Simplifying, we get:

$$2\log_2 x + \frac{\log_2 y}{2} = 7$$
$$\frac{\log_2 8 + \log_2 x}{2} + 1 + 2\log_2 y = 8$$

Continue with the second equation:

$$\frac{3}{2} + \frac{\log_2 x}{2} + 1 + 2\log_2 y = 8$$
$$\frac{\log_2 x}{2} + 2\log_2 y = \frac{11}{2}$$

Let $A = \log_2 x$ and $B = \log_2 y$, and rewrite both equations:

$$2A + \frac{B}{2} = 7$$

$$\frac{A}{2} + 2B = \frac{11}{2}$$

Multiply the second equation through by 4:

$$2A + 8B = 22$$

Now subtract the first equation ($2A + \frac{B}{2} = 7$) from what was just derived to get:

$$\frac{15}{2}B = 15$$

It follows that $B = 2$. Plugging this into $2A + \frac{B}{2} = 7$, we find $A = 3$. Now, solve for x and y: $x = 2^3 = 8$, and $y = 2^2 = 4$. Thus, the desired solution is **(8, 4)**.

3) (8 pts) Create a truth table for the following logical expression: $\neg(p \wedge q) \vee \overline{(\bar{r} \oplus s)}$. Please make sure your first four columns are for p, q, r, and s, respectively, and that you have appropriate intermediary columns.

Solution

p	q	r	s	$\neg(p \wedge q)$	$\bar{r} \oplus s$	$\overline{(\bar{r} \oplus s)}$	$\neg(p \wedge q) \vee \overline{(\bar{r} \oplus s)}$
F	F	F	F	T	T	F	T
F	F	F	T	T	F	T	T
F	F	T	F	T	F	T	T
F	F	T	T	T	T	F	T
F	T	F	F	T	T	F	T
F	T	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	T	T	T	T	T	F	T
T	F	F	F	T	T	F	T
T	F	F	T	T	F	T	T
T	F	T	F	T	F	T	T
T	F	T	T	T	T	F	T
T	T	F	F	F	T	F	F
T	T	F	T	F	F	T	T
T	T	T	F	F	F	T	T
T	T	T	T	F	T	F	F

4) (4 pts) Use the truth table method to prove the following two expressions are logically equivalent:

$$\overline{(p \wedge q)} \rightarrow \bar{q}$$

$$p \vee \bar{q}$$

Solution

p	q	$p \vee \bar{q}$	$\overline{(p \wedge q)}$	\bar{q}	$\overline{(p \wedge q)} \rightarrow \bar{q}$
F	F	T	T	T	T
F	T	F	T	F	F
T	F	T	T	T	T
T	T	T	F	F	T

It can be seen that for all possible settings of p and q, both the column for $p \vee \bar{q}$ and $\overline{(p \wedge q)} \rightarrow \bar{q}$ are identical, so both expressions are logically equivalent to each other.

5) (6 pts) Use the laws of logic to show that two following expressions are logically equivalent:

$$\frac{\overline{(p \wedge q)} \rightarrow \bar{q}}{p \vee \bar{q}}$$

Solution

Expression	Reason
$\overline{(p \wedge q)} \rightarrow \bar{q}$	Given
$\overline{\bar{p} \wedge \bar{q}} \vee \bar{q}$	Implication Identity
$(p \wedge q) \vee \bar{q}$	Double Negation
$\bar{q} \vee (p \wedge q)$	Commutative Law
$(\bar{q} \vee p) \wedge (\bar{q} \vee q)$	Distributive Law
$(\bar{q} \vee p) \wedge T$	Inverse Law
$\bar{q} \vee p$	Identity Law
$p \vee \bar{q}$	Commutative Law

Note: For grading purposes, there's no penalty for skipping the Commutative steps.

6) (5 pts) Use the rules of inference to prove the following argument:

$p \rightarrow r$
 s
 \bar{q}
 $s \rightarrow (p \vee q)$

 $\therefore r$

Solution

Number	Expression	Reason
1	$p \rightarrow r$	Given
2	$\bar{p} \vee r$	Implication Identity with 1
3	s	Given
4	$s \rightarrow (p \vee q)$	Given
5	$p \vee q$	Modus Ponens with 3, 4
6	$q \vee r$	Rule of Resolution with 2, 5
7	\bar{q}	Given
8	r	Disjunctive Syllogism with 6, 7

7) (5 pts) Use the rules of inference to prove the following argument:

$p \rightarrow t$
 \bar{u}
 $\bar{r} \vee s$
 $\bar{q} \rightarrow \bar{t}$
 $u \vee (p \vee r)$

 $\therefore q \vee s$

Solution

Number	Expression	Reason
1	\bar{u}	Given
2	$u \vee (p \vee r)$	Given
3	$p \vee r$	Disjunctive Syllogism with 1, 2
4	$\bar{q} \rightarrow \bar{t}$	Given
5	$\bar{\bar{t}} \rightarrow \bar{q}$	Contrapositive with 4
6	$t \rightarrow q$	Double Negation with 5
7	$p \rightarrow t$	Given
8	$p \rightarrow q$	Law of Syllogism with 7, 6
9	$\bar{r} \vee s$	Given
10	$r \rightarrow s$	Implication Identity with 9
11	$q \vee s$	Constructive Dilemma with 8, 10, 3

8) (5 pts) Prove or disprove the following claim over the universe of all real numbers for x and y.

$$\forall x [\exists y [xy - x - y = -1]]$$

If the claim is false, find a value of x for which it is false. If it is true, show which value(s) of y exist to make the claim true for all x.

Solution

If the equation given is true, we have the following:

$$\begin{aligned}
 xy - x - y &= -1 \\
 xy - x - y + 1 &= 0 \\
 (x - 1)(y - 1) &= 0
 \end{aligned}$$

Either $x = 1$ or $y = 1$. If we consider the case where $y = 1$, then, it doesn't matter what x is set to, the equation is true.

Thus, we can prove the assertion true as follows:

For all x , no matter what x is, consider the case where $y = 1$. Then we have the left hand side equal to:

$$\begin{aligned}xy - x - y &= x(1) - x - 1 \\ &= x - x - 1 \\ &= -1, \text{ RHS}\end{aligned}$$

Thus, we've shown that, for ALL x , there does exist a y (namely, $y = 1$), such that

$$xy - x - y = -1$$

This completes the proof!

9) (5 pts) The last question of each homework assignment will be to write up a two paragraph summary of a topic from the history of mathematics. The idea here is that rarely is any of this history taught in mathematics classes and while I don't have class time to teach it, I thought it would be nice if students learned a bit for each homework assignment. There's no need to use fancy sources, websites will do, but please site which websites you pulled your information from.

Give a summary of the life and mathematical contributions of **Girolamo Cardano**. Note: in some sources, his first name is spelled "Gerolamo."

Sample Write Up

Girolamo Cardano was born in 1501 in Pavia (Italy). Though he had a difficult childhood, he was able to enroll in the University of Pavia in 1520, and after it closed, transferred to the University of Padua, where he earned his doctorate in medicine. He practiced medicine in Saccolongo, but had many talents, and made discoveries in many areas, including mathematics during his life.

In terms of mathematics, Cardano is known for introducing binomial coefficients, publishing work (attributed to others) on the solutions of the cubic and quartic equations, thereby acknowledging the need and existence of imaginary numbers, and publishing one of the first texts on probability. His algebraic achievements were published in his 1545 book, Ars Magna, which discussed negative numbers, imaginary numbers and the solutions provided by Scipione del Ferro to the cubic equation of the form $ax^3 + bx + c = 0$ and his student Lodovico Ferrari's solution to the quartic equation. His book on probability, Liber de Ludo Aleae, was written in the 1500s but wasn't published, posthumously, until 1663 and contains the first formal exposition of probability.

Source: https://en.wikipedia.org/wiki/Gerolamo_Cardano