Important Fall 2022 COT 3100 Final Exam Information

Exam Date and Time
Date: Thursday, December 8, 2022
Time: 10:00 am – 12:50 pm **Location: CB2-106**

NOTE THE EARLIER START TIME!!!

Exam Aids
1. Two sheets of pre-prepared notes, either handwritten or typed (11" x 8.5"), both sides. (Bring This)

2. Calculator (Bring This)

3. Exam #1 Formula Sheet (will be provided)

Exam Format
Part A (25 points): 10:00 am – 10:45 am (ONLY FOR THOSE WHO DID NOT DO COMMUNITY SERVICE. TOPICS: D=RT, Logs, A/G Series, Factorization, Algebra)

Part B (100 points): 10:45 am – 12:50 pm (FOR ALL – Only COT 3100 Syllabus topics: Logic, Sets, Number Theory, Induction + Induction Background Info, Counting, Probability, Relations, Functions)

All Free Response

Rest of Today's Review – Fall 2020 Final Exam, questions not done in 9 am class

COT 3100: Main Message

(1) Appreciate beauty, creativity in math.

(2) Why learn this stuff?

(a) We never know <u>what</u> might be "useful".

- X-ray
- RSA encryption

(b) More commonly, if you understand fundamentals in this course and relate to Algorithms, Data Structs, you'll be able help your workplace solve problems efficiently.

COT 3100 Section 2 Final Exam - Part A (Recitation Topics) - 30 pts

Date: 12/8/2020
Start Time: 1:00 pm EST
End Time: 1:40 pm EST

1) (10 pts) A course grade is comprised of 4 Quizzes, 5 Homework Assignments and 3 Exams. The quizzes are worth 20% of the course grade, the homework assignments are worth 30% of the course grade and the exams are worth the remaining 50% of the course grade. Each of the quizzes is out of 30 points, each of the homework assignments is out of 50 points and each of the exams is out of 100 points. Each quiz is worth the same amount of the course grade, each homework assignment is worth the same amount of the course grade and each exam is worth the same amount of the course grade. Note that a quiz, homework assignment and exam are each worth different amounts of the course grade. Here are Selena's grades on the assignments that have already been turned in:

Quiz 1: 22/30	Homework 1: 45/50	Exam 1: 60/100
Quiz 2: 15/30	Homework 2: 30/50	
Quiz 3: 0/30	Homework 3: 42/50	

As discussed many times in class, Webcourses incorrectly calculates a course grade in the situation when it's known how many assignments in each category there will be for the whole course. We discussed how to properly calculate the course grade in these instances. Thus, as promised, do the following:

(a) Calculate Selena's ACTUAL current course average, which only counts the current assignments, properly weighted.

(b) Selena's grade as it would be calculated on Webcourses.

In order to get any credit on this question, you must show your work and your answers must be perfectly accurate, since incorrect methods will naturally lead to answers that are "pretty close."

2) (10 pts) Let r and s be the roots of the quadratic equation $x^2 - 13x + 9 = 0$. Find the quadratic equation with the roots $2r+s$ and $r + 2s$ with a leading coefficient of 1. (Note: Full credit will only be given for the technique where the value of neither r nor s is ascertained.)

3) (10 pts) For this question, assume that if x_1 ounces of liquid 1 at temperature t_1 is mixed with x_2 ounces of liquid 2 at temperature t_2, the resulting mixture is $x_1 + x_2$ ounces with a temperature of $\frac{t_1x_1+t_2x_2}{x_1+x_2}$. Arup's wife, Anita, likes her tea mixed with milk. (She also likes it with sugar, but I didn't want to make this problem too hard!) Unfortunately, she has a very specific preference for both the ratio of tea to milk and the temperature of the overall mixture. Specifically, she wants a 6:1 ratio of tea to milk at a temperature of 180° F. On Arup's first attempt, he mixed 8 ounces of tea at 200° F with half an ounce of milk at 40° F. It's not possible for Arup to add any more tea to the mixture, but he is allowed to add more milk and he can heat that milk to any temperature greater than 40° F, if necessary, before he adds it. How much milk should Arup add to the tea and at what temperature should he heat it before adding it, to fix the tea?

<u>COT 3100 Section 2 Final Exam - Part B (Logic, Sets, Relations, Functions) - 30 pts</u>

Date: 12/8/2020
Start Time: 1:45 pm EST
End Time: 2:20 pm EST

1) (8 pts) **Use a truth table** to show that the Rule of Resolution on the Rules of Inference list is a one way implication. Namely, prove that the logical expressions to the left and right of the implication sign are **<u>NOT</u>** equivalent. Your table should have seven columns.

2) (7 pts) Let A, B and C be arbitrary sets. **Without the aid of a membership table,** prove:

$$if\ A - B = C - B, then\ A \cup B = C \cup B$$

Note: The proof is long since you have to show that both the LHS is a subset of the RHS and that the RHS is a subset of the LHS. But, the second part of the proof is symmetric to the first (since swapping the roles of A and C doesn't change the problem at all). Thus, just prove the first part and note that the symmetry proves the other subset relation.

3) (10 pts) Let R be the following relation defined over the set of integers:

$R = \{ (a, b) \mid (a + b) \equiv 0 \pmod{10} \}$

Is R reflexive, irreflexive, symmetric, anti-symmetric or transitive?

Please give your answer (yes/no) for each of the five parts and give a brief justification for each. Each answer is worth 1 pt and each justification is worth 1 pt. You can not earn the point for the justification if the answer is incorrect.

4) (5 pts) Let $f(x) = x^3 + ax^2 + bx + c$. It is known that $f(-1) = 84$, $f(0) = 72$ and $f(1) = 50$, what are the values of a, b and c?

COT 3100 Section 2 Final Exam - Part C (Number Theory, Induction) - 30 pts

Date: 12/8/2020
Start Time: 2:25 pm EST
End Time: 3:05 pm EST

Note: Since the answers to the number theory questions are not too hard to guess, ALL of the credit for those questions will be for the work and explanation. Either 0 or 1 point will be awarded for the actual answer. So be very careful to be thorough with your explanations on these questions.

1) (5 pts) Let $n = 2^9k$, where k is a positive odd integer. What percentage of n's divisors are even?

2) (5 pts) Let gcd(a, b) denote the greatest common divisor of positive integers a and b and let lcm(a, b) denote the least common multiple of positive integers a and b. Let a, b and c be arbitrary positive integers. Simplify the following expression so that your final answer is solely in terms of a, b and c, and does not contain any reference to the gcd or lcm of pairs of integers:

$$\frac{\gcd(a,b)\gcd(b,c)\,lcm(a,b)\,lcm(b,c)}{\gcd(a,c)\,lcm(a,c)}$$

3) (10 pts) Let F_n denote the n^{th} Fibonacci number. (Recall $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all integers $n > 2$.) Using induction on n, prove the following summation formula for all positive integers n:

$$\sum_{i=1}^{2n-1} F_iF_{i+1} = F_{2n}^2$$

4) (10 pts) Let a be an odd integer. Prove for all positive integers n via induction that $a^{2^n} - 1$ is divisible by 2^{n+2}.

COT 3100 Section 2 Final Exam - Part D (Counting, Probability) - 35 pts

Date: 12/8/2020
Start Time: 3:10 pm EST
End Time: 3:50 pm EST

1) (5 pts) A subsequence of a sequence is any sequence that can be obtained from the original sequence by deleting 0 or more of the items and keeping all the items in the same order. For example, the sequence A, B, C, D, E, F has the subsequence B, C, F, obtained by deleting A, D and E. How many non-empty subsequences are there of the sequence Q, U, A, K, E, R?

2) (10 pts) Jabba the Hut lives in New York City and loves to eat. We can model New York City as the Cartesian grid. Naturally, each street corner (intersection of a North-South Avenue and East-West Street) is a lattice point on the grid. Jabba lives at (0, 0) and goes to work every morning, which is located at (12, 10), which is 12 blocks east and 10 blocks north of his home. He walks 22 blocks each day to work. (Walking a block means starting at the lattice point (x, y) and either walking east one block to (x+1, y) or north one block to (x, y+1).) Each street corner, except Jabba's home, (0, 0), has a hot dog stand and a pretzel stand. If Jabba arrives at a corner by walking north, he always buys a pretzel at the corner when he arrives, and if he arrives at the corner by walking east, he always buys a hot dog. Jabba has identified that the hot dog stand at (4, 3) serves bad hot dogs. He has also identified that both the pretzel stand AND hot dog stand at (7, 8) serve bad pretzels and hot dogs, respectively. He is determined to avoid all three of these stands while still getting his requisite 22 treats on his way to work. How many different paths can Jabba take to work?

√ 3) (5 pts) A bag contains 6 blue marbles, 7 green marbles and 8 red marbles. If three marbles are chosen out of the bag (without replacement) without looking, what is the probability that all three marbles are blue? Express your answer as a fraction in lowest terms.

√ 4) (10 pts) There are n Christmas lights in a row. Three of these n lights are randomly selected and turned on. The probability that no two of the lights turned on are next to each other is $\frac{39}{60}$. What is n? (Note: You should get a rather ugly quadratic equation when you work this out. Please show all the work to arrive at the quadratic, but use your calculator to find the roots of it.)

5) (5 pts) In what state did the chain restaurant California Pizza Kitchen open its first store?

A3) 6:1 ratio $\quad \frac{6}{1} = \frac{8}{x} \rightarrow x = \frac{4}{3}$ oz

8:x ~~tea~~ tea:milk

Add: $\frac{4}{3} - \frac{1}{2} = \frac{8-3}{6} = \frac{5}{6}$ oz. $\quad 8 + \frac{4}{3} = \frac{28}{3}$

$$\frac{t_1 x_1 + t_2 x_2 + t_3 x_3}{x_1 + x_2 + x_3} = 180°$$

$$\frac{200°(8) + 40°(\frac{1}{2}) + t_3(\frac{5}{6})}{\frac{28}{3}} = 180°$$

$$\begin{array}{r} 1800 \\ -120 \\ \hline 1680 \end{array}$$

$$1600 + 20 + \frac{5t_3}{6} = \cancel{3} \times 180° \times \frac{28}{3}$$

$$1620 + \frac{5t_3}{6} = 60 \times 28$$

$$\frac{5t_3}{6} = 1680 - 1620$$

$$t_3 = \frac{6}{\cancel{5}} \times \cancel{60}^{12} = \boxed{72°}$$

B1)

p	q	r	p ∨ q	p̄ ∨ r	(p∨q)∧(p̄∨r)	q∨r
F	F	F	F	T	F	F
F	F	T	F	T	F	T
F	T	F	T	T	T	T
F	T	T	T	T	T	T
T	F	F	T	F	F	F
T	F	T	T	T	T	T
T	T	F	T	F	F	T
T	T	T	T	T	T	T

B2) Prove if $A-B=C-B$, then $A \cup B \subseteq C \cup B$

it will also serve as proof $C \cup B \subseteq A \cup B$

and we can conclude $A \cup B = C \cup B$.

Direct proof

Let $x \in A \cup B$. Goal: prove $x \in C \cup B$.

$\rightarrow$ Case 1: $x \in B \rightarrow$ by def $\cup$, $x \in C \cup B$ ✓

Case 2: $x \in A \wedge x \in \overline{B}$

We already handled $x \in B$, so assume $x \notin B$.

if $x \in A \wedge x \notin B$, by def $-$, $x \in A-B$

Because $A-B=C-B$, $x \in C-B$.

$\Longrightarrow x \in C \wedge x \notin B$ ✓

By def union, $\boxed{x \in C \cup B}$.

B3) reflexive: no, $(1,1) \notin R$ because $1+1 \equiv 2 \pmod{10}$

irreflexive: no, $(5,5) \in R$ because $5+5 \equiv 10 \equiv 0 \pmod{10}$

symm: yes if $(a,b) \in R$, then $a+b \equiv 0 \pmod{10}$

$b+a \equiv 0 \pmod{10}$

$\Longrightarrow (b,a) \in R$ ✓

anti-symm: no $(3,7) \in R \wedge (7,3) \in R$

transitive: no $(3,7) \in R \wedge (7,3) \in R$ but $(3,3) \notin R$.

B4) $f(0) = \boxed{72 = c}$

$f(1) = 1 + a + b + 72 = 50$

$f(-1) = -1 + a - b + 72 = 84$

$2a + 144 = 134$

$2a = -10$

$\boxed{a = -5}$

$1 - 5 + b + 72 = 50$

$b + 68 = 50$

$\boxed{b = -18}$

$x^3 - 5x^2 - 18x + 72$

C1) Let k have m divisors, then

$n = 2^9 k$ has $(9+1)m = 10m$ divisors

Of these divisors only the ones form $2^0 \times$ whole will be odd $\Rightarrow \frac{m}{10m} = \frac{1}{10}$ odd $\rightarrow \frac{9}{10} = \boxed{90\% \text{ even}}$

C2) $\dfrac{\cancel{gcd}\ \gcd(a,b)\gcd(b,c)\operatorname{lcm}(a,b)\operatorname{lcm}(b,c)}{\gcd(a,c) \times \operatorname{lcm}(a,c)}$

$= \dfrac{ab \cdot bc}{ac} = \boxed{b^2}$

b.c. $n=1$ LHS $= \sum_{i=1}^{2(1)-1} F_i F_{i+1} = F_1 F_2 = 1 \times 1 = 1$ ✓

RHS $= F_{2(1)}^2 = 1^2 = 1$ ✓

I.H. Assume for an arbitrarily chosen positive integer $n=k$ that $\sum_{i=1}^{2k-1} F_i F_{i+1} = F_{2k}^2$

I.S Prove for $n=k+1$ that $\sum_{i=1}^{2(k+1)-1} F_i F_{i+1} = F_{2(k+1)}^2$

$$\sum_{i=1}^{2(k+1)-1} F_i F_{i+1} = \sum_{i=1}^{2k+1} F_i F_{i+1}$$

$$= \sum_{i=1}^{2k-1} F_i F_{i+1} + \overbrace{F_{2k} \cdot F_{2k+1}} + F_{2k+1} \cdot F_{2k+2}$$

$$= F_{2k}^2 + F_{2k} \cdot F_{2k+1} + F_{2k+1} \cdot F_{2k+2} \text{, using IH}$$

$$= F_{2k}\left[F_{2k} + F_{2k+1}\right] + F_{2k+1} \cdot F_{2k+2}$$

$$= F_{2k} \underline{F_{2k+2}} + F_{2k+1} \cdot \underline{F_{2k+2}}$$

$$= F_{2k+2}\left[F_{2k} + F_{2k+1}\right]$$

$$= F_{2k+2} \cdot F_{2k+2} = F_{2k+2}^2$$ ✓

D1) 2^6 subsets of $\{Q, U, A, R, E, R\}$

all can work except empty set...

$\boxed{2^6 - 1}$

D2)

(12,10)

×(7,8) [completely out]

(4,3) [(a) if he gets to 4,3, must go through 4,2)

(0,0) → (4,2)

(b) skip 4,3

(a) Goes (4,2) and (4,3) skips (7,8) (12,10)

$$\binom{6}{2}\left[\binom{15}{8} - \binom{8}{3}\times\binom{7}{2}\right]$$

ways get (4,2)

(4,3) to (7,8)

(b) Paths avoid ~~(4 3) and~~ (4,3) and (7,8).

12 10

$$\binom{22}{10} - \binom{7}{3}\binom{15}{7} - \binom{15}{7}\binom{7}{2} + \binom{7}{3}\binom{8}{3}\binom{7}{2}$$

all

all through (4,3)

all through (7,8)

go through BOTH (4,3) and (7,8)

add answers from (a) and (b)