COT 3100 Final Exam - Part B - 100 pts (12/8/2022) Solutions

1) (5 pts) Use the laws of logic to show that the two following logical expressions are equivalent:

$$(1)((\bar{p} \vee s) \wedge (\bar{s} \vee \bar{p})) \vee (q \wedge (q \vee r)) \qquad\qquad (2) p \rightarrow q$$

Please write each step (including commutative and associative) one by one. More lines have been provided than necessary.

$((\bar{p} \vee s) \wedge (\bar{s} \vee \bar{p})) \vee (q \wedge (q \vee r))$	Given
$((\bar{p} \vee s) \wedge (\bar{p} \vee \bar{s})) \vee (q \wedge (q \vee r))$	Commutative Law
$(\bar{p} \vee (s \wedge \bar{s})) \vee (q \wedge (q \vee r))$	Distributive Law
$(\bar{p} \vee F) \vee (q \wedge (q \vee r))$	Inverse Law
$(\bar{p} \vee F) \vee q$	Absorption Law
$\bar{p} \vee q$	Identity Law
$p \rightarrow q$	Implication Identity

Grading: 1 pt off per error, cap at 5 (commutative step required for full credit), -2 if reasons missing but steps are all correct.

2) (8 pts) UCF is planting several trees in a row on memory mall. Specifically, they've received 10 palm trees, 6 maple trees and 8 pine trees. For aesthetic reasons, the designers have decided that no two palm trees should be planted adjacent to each other. In how many different orders can the trees be planted in a row, with no two palm trees adjacent to each other. (Please leave your answer in combinations/factorials and put a box around your answer.)

Use the other trees (14 of them) as separators. Then there are 15 slots to insert the parm trees (O stands for other):

__ O __ O __ O … O __

We can choose the locations of the palm trees in $\binom{15}{10}$ ways, since we must select exactly 10 of these 15 slots to place a palm tree. For each placement of those, we are free to permute the 6 maples and 8 pines, which can be done in $\frac{14!}{6!8!}$ ways. Since we can pair any of the first selections with any of the second selections, the total number of orderings is $\binom{15}{10}\frac{14!}{6!8!}$.

Grading: 2 pts for mentioning separator idea
2 pts for C(15, 10)
2 pts for arrangement of maple and pine
2 pts to multiply those two

Note: MANY different ways to express this same answer, so be careful!

3) (10 pts) Consider arbitrary sets finite sets A, B, C, D, E and F such that

$$(A \times B) \cup (C \times D) \subseteq E \times F$$

(a) Prove, via proof by contradiction that $A \subseteq E \vee B \subseteq F$.
(b) Give a small example where the given information is true but $A \nsubseteq E$.
(c) Give a small example where the given information is true but $B \nsubseteq F$.

(a) We use proof by contradiction. Assume the opposite that A isn't a subset of E and B isn't a subset of F. This means there exists an element x such that $x \in A \wedge x \notin E$. Similarly, there exists an element y such that $y \in B \wedge y \notin F$. Then, by definition of Cartesian Product, we have that $(x, y) \in A \times B$. By definition of union, we conclude that $(x, y) \in (A \times B) \cup (C \times D)$. Finally, again by definition of Cartesian Product, we have that $(x, y) \notin E \times F$. This contradicts our given information that $(A \times B) \cup (C \times D) \subseteq E \times F$. It follows that our initial assumption must be wrong and we can conclude that $A \subseteq E \vee B \subseteq F$, as desired.

Grading: 1 pt for assuming the opposite of the conclusion.
1 pt for having some element in A not in E
1 pt for having some element in B not in F
1 pt for showing the ordered pair (x,y) (or whatever) is in AxB
1 pt for using that to show it's in AxB U CxD
1 pt for showing that (x,y) isn't in ExF and the conclusion

(b) Let A = {1,2}, B = C = D = ∅, and E = {1} and F = {2}. In this example, the LHS is equal to ∅, since each Cartesian Product is with at least one empty set. The RHS is {(1,2)} but A is NOT a subset of E, since A contains 2 and E does not.

Grading: 1 pt for any specific example, add 1 pt if it's a valid counter ex.

(c) Let B = {1,2}, A = C = D = ∅, and E = {1} and F = {2}. In this example, the LHS is equal to ∅, since each Cartesian Product is with at least one empty set. The RHS is {(1,2)} but B is NOT a subset of F, since B contains 1 and F does not.

Grading: 1 pt for any specific example, add 1 pt if it's a valid counter ex.

4) (10 pts) Find all ordered pairs of integers (x, y) that satisfy the equation 354x + 156y = 78.

Run the GCD algorithm, and then divide each equation through by the GCD:

354 = 2 x 156 + 42
156 = 3 x 42 + 30
42 = 1 x 30 + 12
30 = 2 x 12 + **6 (is gcd)**
12 = 2 x 6

59 = 2 x 26 + 7
26 = 3 x 7 + 5
7 = 1 x 5 + 2
5 = 2 x 2 + 1

Now, run the Extended Euclidean Algorithm on the equations on the right:

5 – 2 x 2 = 1
5 – 2(7 – 5) = 1
3 x 5 – 2 x 7 = 1
3(26 – 3 x 7) – 2 x 7 = 1
3 x 26 – 9 x 7 – 2 x 7 = 1
3 x 26 – 11 x 7 = 1
3 x 26 – 11(59 – 2 x 26) = 1
3 x 26 – 11 x 59 + 22 x 26 = 1
25 x 26 – 11 x 59 = 1

Multiply this equation through by 78 to get:

78 x 25 x 26 – 78 x 11 x 59 = 78
(13 x 25) x 156 – (13 x 11) x 354 = 78
-143 x 354 + 325 x 156 = 78

Thus, one solution is x = -143 and y = 325. It follows that all solutions can be expressed as follows:

$$\{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} = -\mathbf{143} + \mathbf{26}\boldsymbol{c}, \boldsymbol{y} = \mathbf{325} - \mathbf{59}\boldsymbol{c}, \boldsymbol{c} \in \boldsymbol{Z}\}$$

Grading: 3 pts Euclidean
5 pts Extended
1 pt multiply through
1 pt correct offsets for final solution

5) (8 pts) What is the sum of the divisors of 162,000? **<u>Please give your answer in prime factorized form.</u>**

162000 = 162 x 2^3 x 5^3 = 2 x 81 x 2^3 x 5^3 = 2^4 x 3^4 x 5^3

Using the formula from class, the sum of divisors of this number is

$$\frac{2^5-1}{2-1} \times \frac{3^5-1}{3-1} \times \frac{5^4-1}{5-1} = 31 \times \frac{242}{2} \times \frac{624}{4} = 31 \times 121 \times 156 = 31 \times 11^2 \times 4 \times 39$$

$$= 31 \times 11^2 \times 2^2 \times 3 \times 13 = \mathbf{2^2 \times 3 \times 11^2 \times 13 \times 31}$$

Grading: 2 pts prime factorization, 3 pts plugging into sum formulas, 3 pts simplifying into prime factorized form. (Can give partial as you see fit.)

6) (8 pts) Consider arranging the following 15 numbers in a mountain sequence:

1, 2, 3, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 14, 15

A mountain sequence is one where the unique maximum number is neither the first nor last number in the sequence, and all the values from the first to that maximum number are in strictly increasing order and all the numbers that follow the maximum number are in strictly decreasing order. How many different mountain sequences can be formed using all 15 of the numbers listed above? (Note: an example of one such valid sequence is 1, 2, 3, 6, 7, 8, 11, 13, 14, 15, 14, 10, 9, 5, 3.) **<u>Please simplify your answer to a single positive integer (such as 723).</u>**

Place 15 somewhere in the "middle". Since there are 2 copies of 3 and 2 copies of 14, our sequence must look like:

…, 3, …, 14, 15, 14…, 3,…

We must insert the rest of the values: 1, 2, 5, 6, 7, 8, 9, 10, 11, and 13 into this sequence. For every value, we can either choose to insert on the left or on the right of 15. Once we select which relative direction it is (to 15), its location (due to the mountain sequence property) is fixed. Notice that unlike a previous quiz question, there are no restrictions on what goes on which side because we already have items forced to both the left and right of 15. It follows that there are 2^{10} = **<u>1024</u>** mountain sequences with the 15 integers given.

Grading: 1 pt for 15 in middle, 2 pts for 3,14 on both sides
4 pts for recognizing that each of the other 10 items can go on either side
1 pt for final answer
Give 9 out of 10 for the answer $\sum_{i=0}^{10} \binom{10}{i}$.

Note: Partial will be tough here. It'll have to be given on a case by case basis. Other approaches to this problem seem difficult.

7) (10 pts) Let the probability of tossing heads on a biased coin be $\frac{2}{3}$. Consider tossing the coin n times. Let A represent the parity of n (so if n is 10 A is even, if n is 13 A is odd). Prove, using induction on n, that the probability of rolling an A number of heads (so if n is even an even number of heads, if n is odd, an odd number of heads) is $\frac{1+(\frac{1}{3})^n}{2}$. (Note: Unfortunately, this result doesn't hold for probabilities other than $\frac{2}{3}$, as I found out playing around with the problem.)

Base case: n = 1, The probability of getting 1 head (only possible odd number) with 1 toss is $\frac{2}{3}$. Also, note that for n = 1, we have $\frac{1+(\frac{1}{3})^1}{2} = \frac{\frac{4}{3}}{2} = \frac{4}{6} = \frac{2}{3}$. Thus, the base case holds.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer n = k that the probability of getting A number of heads is $\frac{1+(\frac{1}{3})^k}{2}$, where A is even if k is even and A is odd if k is odd.

Inductive Step: Prove for n = k+1 that the probability of getting the parity of k+1 heads on k+1 coin tosses is $\frac{1+(\frac{1}{3})^{k+1}}{2}$.

The probability of getting the same parity of k+1 heads in k+1 tosses is based on the probability of getting the opposite parity (parity of k) number of heads on k tosses. Let A be the parity of k below. In particular, we have:

p(not A # of heads in k+1) tosses) = p(A # of heads in k tosses) x p(heads on last toss) +
p(notA# of heads in k tosses) x p(tails on last toss)

Using the Inductive Hypothesis, that the probability of getting A # of heads in k tosses is $\frac{1+(\frac{1}{3})^k}{2}$, and similarly the probability of getting not A # of heads in k tosses is $1 - \frac{1+\left(\frac{1}{3}\right)^k}{2} = \frac{1-(\frac{1}{3})^k}{2}$, we have:

$$= \frac{(1+\left(\frac{1}{3}\right)^k)}{2} \times \frac{2}{3} + \frac{(1-\left(\frac{1}{3}\right)^k)}{2} \times \frac{1}{3}$$

$$= \frac{\frac{2}{3} + 2\left(\left(\frac{1}{3}\right)^{k+1}\right) + \frac{1}{3} - \left(\frac{1}{3}\right)^{k+1}}{2}$$

$$= \frac{1+\left(\frac{1}{3}\right)^{k+1}}{2}$$

Grading: 1 pt BC, 1 pt IH, 1 pt IS, 2 pts prob formula, 2 pts plug in IH, 1 pt plug probs, 2 pts algebra

8) (10 pts) Let f(x) = ax + b for positive integers a and b with a ≠ 1. In addition, for any function f, define $f^1(n) = f(n)$, and for k > 1, define $f^k(x) = f(f^{k-1}(x))$. Using induction on n, for all positive integers n, prove that $f^n(x) = a^n x + \left(\frac{a^n-1}{a-1}\right)b$.

Base case: n = 1, LHS = $f^1(x) = ax + b$, RHS = $a^1 x + \left(\frac{a^1-1}{a-1}\right)b = ax + b$.
Thus, the given formula holds for n = 1 and the base case is proven.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$f^k(x) = a^k x + \left(\frac{a^k-1}{a-1}\right)b.$$

Inductive Step: Prove for n = k+1 that

$$f^{k+1}(x) = a^{k+1} x + \left(\frac{a^{k+1}-1}{a-1}\right)b.$$

$$f^{k+1}(x) = f(f^k(x))$$
$$= f\left(a^k x + \left(\frac{a^k-1}{a-1}\right)b\right), \text{ via IH}$$

$$= a\left(a^k x + \left(\frac{a^k-1}{a-1}\right)b\right) + b$$

$$= a^{k+1}x + \left(\frac{a(a^k-1)}{a-1}\right)b + \frac{b(a-1)}{a-1}$$

$$= a^{k+1}x + \frac{(a^{k+1} - a + a - 1)}{(a-1)}(b)$$

$$= a^{k+1}x + \frac{(a^{k+1} - 1)}{(a-1)}(b)$$

This completes the proof of the inductive step. It follows that for all positive integers n, $f^n(x) = a^n x + \left(\frac{a^n-1}{a-1}\right)b$.

Grading: 1 pt BC, 1 pt IH, 1 pt IS, 1 pt plug def function comp, 1 pt plug in IH, 2 pts properly apply function f, 3 pts algebra to the end

9) (10 pts) Shelly has 10 different pairs of shoes. She picks eight out of the twenty shoes at random. What is the probability that she picked exactly 3 matching pairs of shoes. Please leave your answer as a fraction with combinations and powers as necessary.

The sample space is $\binom{20}{8}$, since we are choosing 8 shoes out of 20. Now, we must count the number of ways to get exactly 3 matching pairs. We can choose which 3 pairs out of 10 to get in $\binom{10}{3}$ ways. Now, we are left with choosing 2 shoes from the remaining 7 pairs in such a way that we don't get any more pairs. We can select which PAIRS to choose 2 shoes from in $\binom{7}{2}$ ways. Furthermore, once we select each of these pairs, we have 2 selections for which shoe in the pair. Since each of these choices are independent, we multiply to get that there are $\binom{10}{3}\binom{7}{2}2^2$ ways to select 8 shoes with exactly 3 pairs. It follows that our desired probability is:

$$\frac{\binom{10}{3}\binom{7}{2}2^2}{\binom{20}{8}} = \frac{336}{4199}$$

Grading: 2 pts sample space, 2 pts C(10, 3), 2 pts C(7.2), 2 pts 2^2, 2 pts divide

10) (10 pts) Let R be defined over positive integers: R = {(a, b) | 1 < gcd(a, b) < min(a, b)}.

Determine, with proof, whether or not R is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric and (e) transitive. (Note: min(a, b) is the smaller of the two values, a or b.)

R is NOT reflexive since (2, 2) is NOT in R because gcd(2, 2) = 2 and this isn't less than min(2,2).

R IS irreflexive. For all positive integers a, gcd(a, a) = a and min(a, a) = a, so the gcd is not less than the minimum of the two numbers and no ordered pair of the form (a, a) is in R. Thus, R is irreflextive.

R is symmetric. Let (a, b) be an arbitrary element in R. Then, gcd(a, b) = gcd(b, a) < min(b, a), thus (b,a) is also in R.

R is NOT anti-symmetric because (6, 9) and (9, 6) are both in R, since gcd(6, 9) = 3.

R is NOT transitive because (6, 21) is in R and (21, 35) is in R but (6, 35) is not. This is because gcd(6, 21) = 3 and gcd(21, 35) = 7, so while 21 shares a common factor greater than 1 with both 6 and 35, 6 and 35 share no common factors.

Grading: 1 pt for getting each quality (in or out) correct,
1 pt for the reason

11) (10 pts) Given positive integers a and n with a < n and gcd(a, n) = 1, prove that no two items in the set S = $\{ai \mid i \in Z \wedge 1 \leq i \leq n\}$ are equivalent mod n. In your proof you may use the following result: if a | bc and gcd(a, b) = 1, then a | c. Please use proof by contradiction.

Use proof by contraction. Assume that two different values in the set S are equivalent mod n. Let these two values be ai and aj, where i ≠ j and 1 ≤ i, j ≤ n. Then we have:

$$ai \equiv aj \ (mod\ n)$$

$$ai - aj \equiv 0 \ (mod\ n)$$

$$a(i - j) \equiv 0 \ (mod\ n)$$

By definition of mod we have n| (a(i – j)). Since gcd(a, n) = 1, according to the given lemma, it follows that n | (i– j).

But, notice that |i– j| > 0 because i and j are not equal and also notice that |i – j| ≤ n-1, because the difference between the maximum and minimum possible values of i and j is n – 1. This information contradicts our derivation that n | (i – j) because no integers in between 1 and n-1 are divisible by n. It follows that our initial assumption, that two values in the set S were equivalent mod n was incorrect.

Thus, we've proven that all items in the set S are unique, mod n. (Notice that since there are precisely n such items, it follows that there's precisely one value from each equivalence class mod n in the set S.)

Grading: 1 pt basic set up
2 pts having ai and aj being equivalent mod n
1 pt subtract
1 pt factor a
2 pts use lemma to prove a | (i – j)
3 pts to arrive at contradiction based on limits on i and j.

12) (1 pt) Out of what material is Frosty the Snowman made? **SNOW (GIVE TO ALL)**