Exam 2 Thursday

Today is 10/18 Tues
9am

Sheet 1 (30 pts) - Num Theory, Recitation (A.S. / GS
Random Hs)

Sheet 2-3 (total 45 pts) - Induction
20/25
22/23

9am - Sample exam 2 (fall 18)
12pm - = = (fall 20?)

NOTES for 10/18/22 9am

Fall 2018 COT 3100 Exam #2 Make Up

Last Name: ____________________ , **First Name :** ______________________

1) (14 pts) What is the value of the summation below, in terms of n? Simplify your expression to a single polynomial.

$$\sum_{i=n+1}^{2n+1} (3i^2 - 2i)$$

$$\sum_{i=n+1}^{2n+1} (3i^2-2i) = \left(\sum_{i=1}^{2n+1} 3i^2 - \sum_{i=1}^{n} 3i^2\right) - \left(\sum_{i=1}^{2n+1} 2i - \sum_{i=1}^{n} 2i\right)$$

$$= \frac{\cancel{3}(2n+1)(\underset{n+1}{\cancel{2n+2}})(4n+3)}{\cancel{6}} - \frac{\cancel{3}n(n+1)(2n+1)}{\cancel{6}\,2}$$

$$- \frac{\cancel{2}(2n+1)(2n+2)}{\cancel{2}} + \frac{\cancel{2}n(n+1)}{\cancel{2}}$$

$$= (2n+1)(n+1)(4n+3) - \frac{n(n+1)(2n+1)}{2} - (2n+1)(2n+2) + n(n+1)$$

$$= \frac{(n+1)}{2}\left[2(2n+1)(4n+3) - n(2n+1) - 2(2n+1)\overset{2}{\cancel{(2n+2)}} + 2n\cancel{(n+1)}\right]$$

$$= \frac{(n+1)}{2}\left[16n^2 + 20n + 6 - 2n^2 - n - \cancel{8n^2 - 12n - 4 + 2n^2 + 2n} - 8n - 4 + 2n\right]$$

$$= \boxed{\frac{(n+1)}{2}\left[\cancel{16n^2}\ 14n^2 + 13n + 2\right]}$$

$$= \frac{(n+1)(14n^2+13n+1)}{2}$$

2) (14 pts) Prove using induction on n that for all positive integers n, $81 \mid (10^n - 9n - 1)$.

base case $n=1$ $\quad 10^1 - 9(1) - 1 = 10 - 9 - 1 = 0$

$0 = 81 \times 0$, so $81 \mid 0$, and the base case holds

Inductive hypothesis: Assume for an arbitrarily chosen positive integer $n=k$ that $81 \mid (10^k - 9k - 1)$.

$\exists c \in \mathbb{Z} \mid 10^k - 9k - 1 = 81c$.

Inductive Step: Prove for $n = k+1$ that

$$81 \mid (10^{k+1} - 9(k+1) - 1)$$

$$\begin{aligned}
10^{k+1} - 9(k+1) - 1 &= 10 \times 10^k - 9k - 9 - 1 \\
&= \underline{10} \times \underline{10^k} - 9k - \underline{10} \times \underline{1} \\
&= 10 \times 10^k - 9k - 81k - 10 \times 1 + 81k \\
&= 10 \times 10^k - 90k - 10 + 81k \\
&= 10(10^k - 9k - 1) + 81k \\
&= 10(81c) + 81k, \text{ using I.H.} \\
&= 81(10c + k)
\end{aligned}$$

Since $c, k \in \mathbb{Z}$, $10c + k \in \mathbb{Z}$. It follows that $81 \mid 10^{k+1} - 9(k+1) - 1$.

This proves the inductive step. Thus we can conclude for all pos. ints. n, $81 \mid (10^n - 9(n) - 1)$.

3) (15 pts) Determine $175^{-1} \bmod 216$.

$216 = 1 \times 175 + 41$
$175 = 4 \times 41 + 11$
$41 = 3 \times 11 + 8$
$11 = 1 \times 8 + 3$
$8 = 2 \times 3 + 2$
$3 = 1 \times 2 + 1$

$3 - 1 \times 2 = 1$
$3 - (8 - 2 \times 3) = 1$
$3 \times 3 - 1 \times 8 = 1$
$3(11 - 8) - 1 \times 8 = 1$
$3 \times 11 - 4 \times 8 = 1$
$3 \times 11 - 4(41 - 3 \times 11) = 1$
$3 \times 11 - 4 \times 41 + 12 \times 11 = 1$
$15 \times 11 - 4 \times 41 = 1$
$15(175 - 4 \times 41) - 4 \times 41 = 1$
$15 \times 175 - 60 \times 41 - 4 \times 41 = 1$
$15 \times 175 - 64 \times 41 = 1$
$15 \times 175 - 64(216 - 175) = 1$
$15 \times 175 - 64 \times 216 + 64 \times 175 = 1$
$79 \times 175 - 64 \times 216 = 1$ take mod 216
$\underline{79} \times \underline{175} - 64 \times 0 \equiv 1$ mod 216

$$\boxed{175^{-1} \equiv 79 \bmod 216}$$

4) (10 pts) Find the ***product of the divisors*** of 40000, leaving your answer in ***prime factorized form.***

$40000 = 4 \times 10^4 = 2^2 \times 2^4 \times 5^4 = 2^6 \times 5^4$

has $(6+1)(4+1) = 7 \times 5 = 35$ divisors

17 pairs multiply to 40000

1 divisor 200 (on its own)

$$\text{Product} = (40000)^{17} \cdot 200$$

$$= (2^6 \times 5^4)^{17}(2^3 \times 5^2) = 2^{102} \times 5^{68} \times 2^3 \times 5^2$$

$$= \boxed{2^{105} \times 5^{70}}$$

5) (12 pts) Let $a_1, a_2, a_3, \ldots$ form an infinite geometric sequence with $a_3 = 9$ and $a_6 = \frac{8}{3}$. Determine the following sum: $\sum_{i=1}^{\infty}(a_{2i-1} - a_{2i})$.

$a_6 = a_3 r^3$

$\frac{8}{3} = 9 \times r^3$

$r^3 = \frac{8}{27}$

$\rightarrow \boxed{r = \frac{2}{3}} \Rightarrow r^2 = \frac{4}{9}$

$a_1 = 9 \times \frac{3}{2} \times \frac{3}{2} = \frac{81}{4}$

$a_2 = \frac{27}{2}$

Sol 1

ratio $\frac{4}{9}$

$$\sum_{i=1}^{\infty} a_{2i-1} - \sum_{i=1}^{\infty} a_{2i}$$

$$\frac{81/4}{1 - \frac{4}{9}} - \frac{27/2}{1 - \frac{4}{9}}$$

$$= \frac{9}{5}\left(\frac{81}{4} - \frac{54}{4}\right) = \cancel{\boxed{\frac{81}{4}}}$$

$$\frac{243}{20}$$

Sol 2

$$\sum_{i=1}^{\infty} a_{2i-1} - a_{2i}$$

$$= \sum_{i=1}^{\infty} a_{2i-1}(1-r)$$

$$= \left(1 - \frac{2}{3}\right)\sum_{i=1}^{\infty} a_{2i-1}$$

$$= \frac{1}{3}\left[\frac{\frac{81}{4}}{1 - \frac{4}{9}}\right]$$

$$= \cancel{\boxed{\frac{81}{4}}} \quad \frac{27}{4} \times \frac{9}{5} = \boxed{\frac{243}{20}}$$

6) (10 pts) Let t_n be defined as follows: $t_0 = -2$, $t_1 = 4$, $t_n = 8t_{n-1} - 16t_{n-2}$, for all integers $n \geq 2$. Prove, using strong induction on n, that for all non-negative integers n, $t_n = (3n-2)4^n$.

base cases $n=0$ $LHS = t_0 = -2$

$RHS = (3(0)-2)\cdot 4^0 = -2 \times 1 = -2$

$n=1$ $LHS = t_1 = 4$

$RHS = (3\cdot 1-2)\cdot 4^1 = 1\cdot 4^1 = 4$

Assertion is true for $n=0$ and $n=1$.

Inductive hypothesis: Assume for all non-negative integers n, $0 \leq n \leq k$, where k is an arbitrarily chosen positive integer that

~~$t_k = (3k-2)$~~ $t_n = (3n-2)4^n$.

Inductive step: Prove for $n = k+1$ that

$t_{k+1} = (3(k+1)-2)4^{k+1}$

$$
\begin{aligned}
t_{k+1} &= 8t_k - 16t_{k-1} \\
&= 8(3k-2)4^k - 16(3k-5)4^{k-1} \\
&= 2(3k-2)4^{k+1} - (3k-5)4^{k+1} \\
&= 4^{k+1}[6k - 4 - 3k + 5] \\
&= 4^{k+1}[3k+1] \\
&= \boxed{4^{k+1}(3(k+1)-2)}
\end{aligned}
$$

This proves the inductive step. It follows that for all non-neg ints n, $t_n = (3n-2)4^n$.

nothing to do w/ base case!

7) (10 pts) Let a be a positive real number with a ≥ 2. Using induction on n, prove for all non-negative integers n that

$$\sum_{i=0}^{n} a^i < \frac{a^{n+1}}{a-1}$$

base case n=0 $\quad LHS = \sum_{i=0}^{0} a^i = a^0 = 1$

$$RHS = \frac{a^{0+1}}{a-1} = \frac{a}{a-1} = 1 + \frac{1}{a-1}$$

Because $a \geq 2$, LHS < RHS.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n=k that $\sum_{i=0}^{k} a^i < \frac{a^{k+1}}{a-1}$

Inductive Step: Prove for n=k+1 that

$$\sum_{i=0}^{k+1} a^i < \frac{a^{k+2}}{a-1}$$

$$\sum_{i=0}^{k+1} a^i = \sum_{i=0}^{k} a^i + a^{k+1}$$

$$< \frac{a^{k+1}}{a-1} + \frac{a^{k+1}}{(a-1)} \times (a-1), \text{ using IH}$$

$$= \frac{a^{k+1} + a^{k+2} - a^{k+1}}{a-1}$$

$$= \boxed{\frac{a^{k+2}}{a-1}} \checkmark$$

This proves the inductive step. It follows for all non-neg ints n and real # $a \geq 2$ $\sum_{i=0}^{n} a^i < \frac{a^{n+1}}{a-1}$

8) (13 pts) Let H_n denote the n[th] Harmonic number. Recall that $H_n = \sum_{i=1}^{n} \frac{1}{i}$. Also, note that $\ln(n+1) - \ln n \geq \frac{1}{n+1}$. For all positive integers n, prove that $H_n \leq \ln(n) + 1$.

$\ln n \leq \ln(n+1) - \frac{1}{n+1}$

base case $n = 1$ $\quad LHS = H_1 = 1$

$RHS = \ln(1) + 1 = 1$

Assertion holds for $n = 1$.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer $n = k$ that $H_k \leq \ln(k) + 1$

Inductive step: Prove for $n = k+1$ that $H_{k+1} \leq \ln(k+1) + 1$

$$H_{k+1} = H_k + \frac{1}{k+1}$$

$$\leq \ln(k) + 1 + \frac{1}{k+1}, \text{ using I.H}$$

$$\leq \ln(k+1) - \frac{1}{k+1} + 1 + \frac{1}{k+1}, \text{ using hint}$$

$$= \ln(k+1) + 1$$

This proves the inductive step. Thus we can conclude for all positive integers n that

$$H_n \leq \ln(n) + 1$$

9) (2 pts) Which office supply company bought the naming rights to the Staples Center in LA?

____________ STAPLES