COT 3100 10/18/22 12pm

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1. talk Random Algebra Quiz
2. Exam 2 Review

Problem A: $a_{15} = 12$, $a_{20} = 32$ Find a_{100}

Problem B: $a_{15} = 2n+4$, $a_{20} = 17n-16$ Find a_{100}

$d = \frac{a_{20} - a_{15}}{5} = \frac{32-12}{5} = 4$ $a_{100} = a_{20} + 80d$

$a_{100} = 32 + 80(4)$

$d = \frac{a_{20} - a_{15}}{5} = \frac{(17n-16)-(2n+4)}{5} = 3n-4$, $a_{100} = 17n-16 + 80(3n-4)$

* Don't be afraid of letters!

Know when to treat something like a const.

Odd terms in inf. geo sequence have ratio r^2 when whole seq has ratio r.

$a_1 + a_1 r^2 + a_1 r^4 + \dots$

$a_2 + a_4 + a_6$

$a_1 r + a_3 r + a_5 r + \dots = r(a_1 + a_3 + \dots)$

Sum even terms = $r \times$ (Sum Odd Term)

Let x = amt of time both paint.

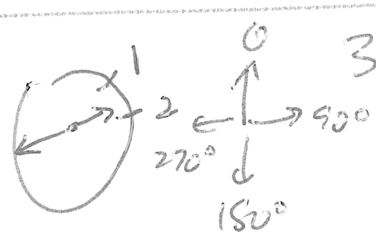
Jenny paint $\frac{x}{4}$ walls

John paints $\frac{x}{5}$ walls

$$\frac{x}{4} + \frac{x}{5} = 8$$

$$\frac{5x + 4x}{20} = 8$$

* Don't be afraid of letters!!!
(clearly delineate a variable to represent a concept!)



360° circle

$$\frac{360°}{12} = 30° \text{ hr hand/hr}$$

360° degree/hr

hr hand $\frac{30°}{60} = \frac{1}{2}°/\text{min}$

min hand $\frac{360°}{60} = 6°/\text{min}$

$$\underbrace{30° + \frac{t}{2}}_{\text{hour}} + \underbrace{180°}_{\text{opposite}} = \underbrace{6t}_{\text{min}}$$

Exam 2 Review

~~Page~~/Sheet 1 - Num Theory + Recitation Problems (30 pts)

base conv, division, Euclid's, <u>Linear Eqn Solver</u> + mod inverse, prime fact, # divisors, Σ divisors, Algebra + Arith/Geo Series

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Sheets 2+3 - Induction (45 pts)

summation =, <=

divisibility

Creative

Something like one of the others (recurrence relations, matrices, etc.)

Fall 2019 COT 3100 Exam #2 (10/24/2019) (Note: Out of 100 points) - Pages 1, 2

Last Name: ____________________ , **First Name :** ______________________

Lab Section: 18(R9) 19(R10) 20(R11) 21(T2) 22(T3) 23(T4) 24(T5)

1) (10 pts) Eileen rides her bike a distance of D miles at a rate of r miles per hour from home to the bus stop. She then takes the bus for a distance of 5D miles traveling at a rate of 4r miles per hour to arrive at work. Assume that the bus starts immediately after Eileen arrives to the bus stop. If the average speed for Eileen's trip from home to work is 32 miles per hour, at what rate, in miles per hour, does she ride her bike for that portion of the trip?

$$\text{Distance} = D + 5D = 6D$$

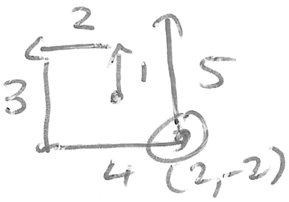
$$\text{time} = \frac{D}{r} + \frac{5D}{4r} = \frac{4D}{4r} + \frac{5D}{4r} = \frac{9D}{4r}$$

$$32 \text{ mph} = \frac{6D}{\frac{9D}{4r}}$$

$$32\text{mph} = \frac{24r}{9}$$

$$r = \frac{9 \times \cancel{32}^{4}}{\cancel{24}_{3}} \text{ mph} = \boxed{12 \text{ mph}}$$

2) (12 pts) Consider a particle on the Cartesian plane that starts at point (0, 0) at time t=0, initially headed in the direction of the positive y-axis. At time t, the particle will move t+1 units in the direction that it is headed, with the first movement starting at t=0. Thus, at t=1, the particle will end up at (0,1). After each move, the particle will turn left 90 degrees, instantaneously. Thus, at t=2, the particle will end up at (-2, 1), since it's headed in the direction of the negative x-axis and travels 2 units in that direction. Using induction on n, prove for all non-negative integers n, that at time t = 4n, the particle will be located at (2n, -2n), heading in the positive y-axis.



base case: $n=0$ at $t=0$ particle $(0,0)$
pt $(2(0), -2(0)) = (0,0)$
The assertion is true for $n=0$.

Inductive Hypothesis: Assume for an arbitrary non-neg int $n=k$ that at time $4k$, the particle will be at $(2k, -2k)$.

Inductive step: Prove for $n=k+1$ that at time $4(k+1) = 4k+4$, the particle will be at $(2(k+1), -2(k+1)) = (2k+2, -2k-2)$.

Where will particle be at $t = 4k+4$?
By I.H. particle is at $(2k, -2k)$ at $t=4k$. It rotates $360°$ each 4 seconds so it direction is in the positive y-axis

Trace ~~partice~~ particle: at $t = 4k+1$ particle moves up $4k+1$ units to $(2k, -2k+4k+1) = (2k, 2k+1)$.
Turn left at $t = 4k+2$ $(2k-(4k+2), 2k+1) = (-2k-2, 2k+1)$
Turn left at $t = 4k+3$ go down $4k+3 \Rightarrow (-2k-2, 2k+1 - 4k-3)$
$= (-2k-2, -2k-2)$

Turn left at $t = 4k+4$ go forward $4k+4$ arrive at ✓
$(-2k-2+4k+4, -2k-2) = (2k+2, -2k-2)$

Fall 2019 COT 3100 Exam #2 (10/24/2019) (Note: Out of 100 points) - Pages 3, 4

Last Name: ___________________ , First Name : ______________________

Lab Section: 18(R9) 19(R10) 20(R11) 21(T2) 22(T3) 23(T4) 24(T5)

3) (15 pts) Find all integer solutions to the equation 258x + 204y = 42.

$258 = 1\times204 + 54$
$204 = 3\times54 + 42$
$54 = 1\times42 + 12$
~~$42 = 3\times14$~~
$42 = 3\times12 + \boxed{6}$
$12 = 2\times6$
gcd = 6

Divide through by gcd

$43x + 34y = 7$

$43 = 1\times34 + 9$
$34 = 3\times9 + 7$
$9 = 1\times7 + 2$
$7 = 3\times2 + 1$

Run Extended Euclidean

$7 - 3\times2 = 1$
$7 - 3(9-7) = 1$
$7 - 3\times9 + 3\times7 = 1$
$4\times7 - 3\times9 = 1$
$4(34 - 3\times9) - 3\times9 = 1$
$4\times34 - 12\times9 - 3\times9 = 1$
$4\times34 - 15\times9 = 1$
$4\times34 - 15(43-34) = 1$
$4\times34 - 15\times43 + 15\times34 = 1$
$(19\times34 - 15\times43 = 1)$
$x = -15$, $y = 19$ is one sol to = 1

mult by 7

$(7\times19)34 - (7\times15)43 = 7$
$133\times34 - 105\times43 = 7$
$x = -105$, $y = 133$ is one solution

All solutions = $\{(x,y) \mid x = -105 + 34c, y = 133 - 43c, c \in \mathbb{Z}\}$

4) (10 pts) Let x and y be integers such that 19 | (5x - y). Prove that 19 | (x+15y).

Given $\exists c \in \mathbb{Z} \mid 5x - y = 19c$

$\Rightarrow y = 5x - 19c$

$$x + 15y = x + 15(5x - 19c)$$
$$= x + 75x - 15 \times 19c$$
$$= 76x - 19 \times (15c)$$
$$= 19(4x - 15c)$$

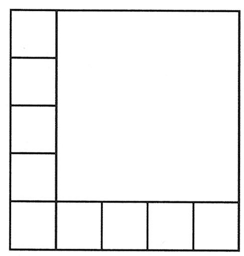
because $x, c \in \mathbb{Z}$, $4x - 15c \in \mathbb{Z}$ and we've proven that $19 \mid (x+15y)$

Fall 2019 COT 3100 Exam #2 (10/24/2019) (Note: Out of 100 points) - Pages 5, 6

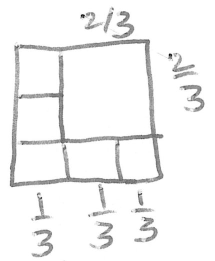
Last Name: ___________________, **First Name :** ____________________

Lab Section: 18(R9) 19(R10) 20(R11) 21(T2) 22(T3) 23(T4) 24(T5)

5) (15 pts) Using strong induction on n with three base cases, prove that a square can be partitioned into n squares, for all positive integers $n \geq 6$. (Note: To partition a square, you must draw some line segments dividing the square so that all of the separate pieces are non-intersecting, cover the whole square, and are all squares themselves. The picture below shows a square partitioned into 10 squares. Specifically in this picture, if the original square has side length s, 9 of the squares have side length $\frac{s}{5}$, and the tenth square in the upper right hand corner has side length $\frac{4s}{5}$.) Your proof should primarily have pictures accompanied with the specific dimensions of each of the smaller squares in terms of the square to be partitioned.

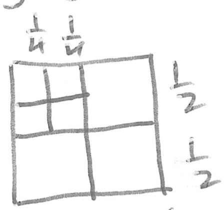


base cases n=6



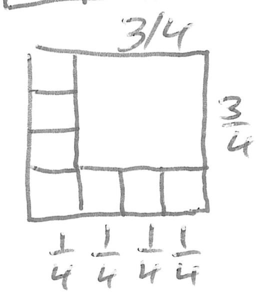
Split square into 6 squares
5 of size $\frac{1}{3}$ by $\frac{1}{3}$
1 of size $\frac{2}{3}$ by $\frac{2}{3}$ as shown.

n=7



Split square into 7 squares
4 of size $\frac{1}{4}$ by $\frac{1}{4}$
3 of size $\frac{1}{2}$ by $\frac{1}{2}$

n=8

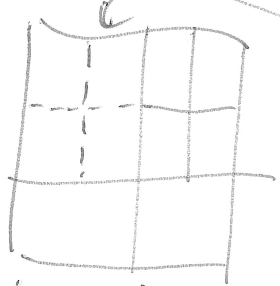


Split square into 8 squares
7 of size $\frac{1}{4}$ by $\frac{1}{4}$
1 of size $\frac{3}{4}$ by $\frac{3}{4}$

Inductive hypothesis: Assume for all integers n, $6 \leq n \leq k$, where k is arbitrarily chosen and greater than or equal to 8, that we can partition a square into n squares.

Inductive Step: Prove for $k+1$ that we can partition a square into $k+1$ squares

Using the I.H. because $k \geq 8$, $k+1 \geq 9$ and $k-2 \geq 6$, we can partition a square into $k-2$ squares. Take that partition, pick any square in it, ~~and~~ of size s, and split that into 4 squares of size $s/2$. See dotted line



Now we have a new partition with $(k-2)+4-1 = k+1$ squares as desired.

prime number ok!

6) (14 ~~pts~~) Let p be an ~~positive integer~~ (greater than 1) such that $2^p - 1$ is a prime number. Let $n = 2^{p-1}(2^p-1)$. Determine the sum of divisors of n, in terms of either n, p or both. **(Note: Full credit is given only for the most simple form of the final answer.)**

is prime factorization

all divisors are of form $2^a \cdot (2^p-1)^b$

$0 \le a \le p-1$

$0 \le b \le 1$

first add all divisors w/ $b = 0$

$$\underbrace{2^0 + 2^1 + 2^2 + \cdots + 2^{p-1} = \boxed{2^p - 1}}$$

next add all divisors w/ $b = 1$

$$(2^p-1)\cdot 2^0 + (2^p-1)\cdot 2^1 + (2^p-1)\cdot 2^2 + \cdots 2^{p-1}(2^p-1)$$

$$= (2^p-1)\left[2^0 + 2^1 + \cdots + 2^{p-1}\right]$$

$$= \underbrace{(2^p-1)(2^p-1)}$$

$$(2^p-1) + (2^p-1)(2^p-1)$$

$$= (2^p-1)(1 + 2^p - 1)$$

$$= \boxed{2^p(2^p-1)}$$

twice n. and Subtract n and sum of proper divisors equals n!

$n = 2^2 \cdot (2^3-1) = 28$

$1 + 2 + 4 + 7 + 14 = 28$ ✓

Perfect Numbers!

Mersenne Primes

No Odd Perfect #s.

Unknown whether there are an infinite # of these

Fall 2019 COT 3100 Exam #2 (10/24/2019) (Note: Out of 100 points) - Pages 7,8

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7) (12 pts) Using induction on n, prove for all non-negative integers n that $\sum_{i=0}^{n}\frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

base case $n=0$ $\quad LHS = \sum_{i=0}^{0}\frac{i}{2^i} = \frac{0}{2^0} = 0$ ✓

$RHS = 2 - \frac{0+2}{2^0} = 2-2 = 0$ ✓

Inductive Hypothesis: Assume for an arbitrarily chosen non-neg int $n=k$ that $\sum_{i=0}^{k}\frac{i}{2^i} = 2 - \frac{k+2}{2^k}$

Inductive Step: Prove for $n=k+1$ that $\sum_{i=0}^{k+1}\frac{i}{2^i} = 2 - \frac{k+3}{2^{k+1}}$

$$\sum_{i=0}^{k+1}\frac{i}{2^i} = \sum_{i=0}^{k}\frac{i}{2^i} + \frac{k+1}{2^{k+1}}$$

$$= 2 - \frac{(k+2)2}{2\cdot 2^k} + \frac{k+1}{2^{k+1}}, \text{ using I.H.}$$

$$= 2 - \frac{[2(k+2)-(k+1)]}{2^{k+1}}$$

$$= 2 - \frac{[2k+4-k-1]}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}}$$ ✓

8) (10 pts) On Bobby's first day of work, he earns \$100. On every subsequent day, he earns \$4 more than the previous day. On Selena's first day of work, she earns \$10. On every subsequent day, she earns \$6 more than the previous day. Both start work on the same day. After n days of both working, they compare their ***total*** earnings to that point in time and realize that Selena has earned ***strictly more money*** in total than Bobby. What is the minimum possible value of n?

Bobby: 100, 104, 108, etc. $a_1 = 100 \quad d = 4, \quad a_n = 100 + 4(n-1)$

Selena: 10, 16, 22, etc. $a_1' = 10 \quad d' = 6, \quad a_n' = 10 + 6(n-1)$

$$S_n = \frac{(100 + 100 + 4n - 4)}{2} \times n = \frac{(4n + 196)}{2} \times n = (2n+98)n$$

$$S_n' = \frac{(10 + 10 + 6n - 6)}{2} \times n = \frac{(6n + 14)}{2} \times n = (3n+7)n$$

$$(3n+7)n > (2n+98)n$$

$3n+7 > 2n+98$, since $n > 0$.

$$n > 91$$

minimum value of n is $\boxed{92}$

9) (2 pts) By what acronym is the National Basketball Association typically referred? NBA