COT 3100 Exam #2 - Solutions (10/20/2022)

1) (12 pts) Find all ordered pairs of integer solutions (x, y), to the equation

$$309x + 83y = 12$$

To receive credit, you must use the process shown in class.

Run the Extended Euclidean algorithm on a = 309, b = 83:

309 = 3 x 83 + 60
83 = 1 x 60 + 23
60 = 2 x 23 + 14
23 = 1 x 14 + 9
14 = 1 x 9 + 5
9 = 1 x 5 + 4
5 = 1 x 4 + 1

5 – 1 x 4 = 1
5 – (9 – 5) = 1
2 x 5 – 1 x 9 = 1
2(14 – 9) – 1 x 9 = 1
2 x 14 – 2 x 9 – 1 x 9 = 1
2 x 14 – 3 x 9 = 1
2 x 14 – 3(23 – 14) = 1
2 x 14 – 3 x 23 + 3 x 14 = 1
5 x 14 – 3 x 23 = 1
5(60 – 2 x 23) – 3 x 23 = 1
5 x 60 – 10 x 23 – 3 x 23 = 1
5 x 60 – 13 x 23 = 1
5 x 60 – 13(83 – 60) = 1
5 x 60 – 13 x 83 + 13 x 60 = 1
18 x 60 – 13 x 83 = 1
18(309 – 3 x83) – 13 x 83 – 1
18 x 309 – 54 x 83 – 13 x 83 = 1
18 x 309 – 67 x 83 = 1

Take last step on left and multiply through by 12:

(12 x 18) x 309 + (-12 x 67) x 83 = 1 x 12

One solution for x = 12 x 18 = 216.
One solution for y = -12 x 67 = -804.

All solutions are expressed by the following set:
{ (x, y) | x = 216 + 83c, y = -804-309c | c ∈ Z }

An equivalent "centered" representation of this set is **{ (x, y) | x = 50 + 83c, y = -186-309c | c ∈ Z }**

Grading: 3 pts Euclidean, 6 pts Extended Euclidean, 1 pt mult 12, 2 pts for final solution with correct offsets

Notes: If someone only forgot the multiply by 12 step and got x = 18 + 83c, y = -67-309c, award 11/12.

2) (8 pts) The community of Celebration is unique in that all of its houses are identical. Johnny, Selma and Malik are painting houses in Celebration. It takes Johnny 10 days to paint a house in Celebration by himself, it takes Selma 6 days to paint a house in Celebration by herself, and it takes Malik 4 days to paint a house in Celebration by himself. Assume all 3 paint at a constant rate. How long, **in days**, will it take the three of them working together to paint 31 Celebration houses?

Let x equal the number of days the three paint. Then in x days they collectively paint $\frac{x}{10} + \frac{x}{6} + \frac{x}{4}$ houses. We are given that this quantity is 31, so set these two expressions equal to each other and solve for x:

$$\frac{x}{10} + \frac{x}{6} + \frac{x}{4} = 31$$

$$\frac{6x + 10x + 15x}{60} = 31$$

$$\frac{31x}{60} = 31$$

$$x = \mathbf{60}$$

Thus, the three will complete the task in 60 days. (Grading: 3 pts initial equation, 5 pts solve, give partial as you see fit.)

3) (5 pts) What is the remainder when 7^{22} is divided by 11? Use any method you'd like and clearly show your work.

Let's use the bottom up fast exponentiation method taught in class, and build a table of modular exponents upto 7^{16}:

Exp	1	2	4	8	16
7^{exp} mod 11	7	5	3	9	4

$7^{22} = 7^{16}7^{4}7^{2} \equiv 4(3)(5) \equiv 60 \equiv 5 \bmod 11$

The desired remainder is 5. (Grading: 5 pts correct answer, Take off 1 pt per error if a valid approach is used but the answer is not correct.)

4) (5 pts) X is a 3 digit number, when represented in base 10. If X is represented in base 39, it ends in a 0. If X is represented in base 45, it ends in a 0. What is the value of X (in base 10)?

This means that X is divisible by 39 and 45. By definition, it must be divisible by the least common multiple of 39 and 45. We can either run the Euclidean Algorithm, or quickly inspect that 39 = 3 x 13 and 45 = 3^2 x 5 to find that the gcd(39, 45) = 3. It follows that $lcm(39,45) = \frac{39\times45}{3} = 13 \times 45 = 450 + 135 = \mathbf{585}$. No other multiple of 585 is 3 digits so this is the answer.

Grading: 3 pts LCM observation, 2 pts LCM calculation. Give 4/5 for answer 195.

5) (10 pts) Using induction on n, prove for all positive integers n that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Base case: n = 1, LHS = $\sum_{i=1}^{1} \frac{1}{i(i+1)(i+2)} = \frac{1}{1\times 2\times 3} = \frac{1}{6}$, RHS = $\frac{1(1+3)}{4(1+1)(1+2)} = \frac{4}{24} = \frac{1}{6}$.

Thus, the assertion is true for n = 1. **(1 pt)**

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer n = k that: **(1 pt)**

$$\sum_{i=1}^{k} \frac{1}{i(i+1)(i+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$$

Inductive Step: Prove for n = k+1 that: **(1 pt)**

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)(i+2)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$\sum_{i=1}^{k+1} \frac{1}{i(i+1)(i+2)} = \left[\sum_{i=1}^{k} \frac{1}{i(i+1)(i+2)}\right] + \frac{1}{(k+1)(k+2)(k+3)}$ **(1 pt)**

$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$, using IH **(2 pts)**

$= \frac{k(k+3)(k+3)}{4(k+1)(k+2)(k+3)} + \frac{4}{4(k+1)(k+2)(k+3)}$

$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$

$= \frac{(k+1)(k+1)(k+4)}{4(k+1)(k+2)(k+3)}$

$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$ (**Algebra to end: 4 pts)**

This completes the inductive step. It follows that for all positive integers n,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

6) (12 pts) Prove, using induction on n, that for all non-negative integers n that $16 \mid (15^n - 7^n + 8n)$.

Base case: n = 0, Expression = $15^0 - 7^0 + 8(0) = 1 - 1 + 0 = 0$.
Note that $16 \mid 0$, thus the assertion is true for n = 0. **(1 pt)**

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer n = k that $16 \mid (15^k - 7^k + 8k)$. Namely, there exists an integer c such that $16c = 15^k - 7^k + 8k$. **(1 pt)**

Inductive Step: Prove for n = k+1 that $16 \mid (15^{k+1} - 7^{k+1} + 8(k+1))$. **(1 pt)**

$15^{k+1} - 7^{k+1} + 8(k+1) = 15 \times 15^k - 7 \times 7^k + 8k + 8$ **(2 pts)**

$= 8 \times 15^k + 7 \times 15^k - 7 \times 7^k + 8k + 48k - 48k + 8$

$= (7 \times 15^k - 7 \times 7^k + 56k) - 48k + 8 \times 15^k + 8$

$= 7(15^k - 7^k + 8k) - 48k + 8(15^k + 1)$ (**Isolate IH: 3 pts)**

$= 7(16c) - 48k + 8(15^k + 1)$, using IH **(Use IH – 2 pts)**

Note that 15^k is odd for all non-negative integers k because 1 is odd and any number of products of odd integers is odd. It follows that there exists an integer d such that $15^k = 2d + 1$. **(1 pt)**

$= 7(16c) - 48k + 8((2d + 1)+ 1)$

$= 7(16c) - 48k + 8(2d + 2)$

$= 7(16c) - 48k + 16(d+1)$

$= 16(7c + 3k + d + 1)$ **(1 pt)**

Because c, d and k, are integers, 7c + 3k + d + 1 is also an integer. It follows that $15^{k+1} - 7^{k+1} + 8(k+1)$ is divisible by 16, proving the inductive step. We can conclude for all non-negative integers, n, that $16 \mid (15^n - 7^n + 8n)$.

7) (10 pts) Prove, using strong induction on n, that the equation $4x + 5y + 7z = n$ always has at least one non-negative integer ordered triplet (x, y, z) as a solution for all integers $n \geq 7$. You will have to use **four** base cases.

Base cases: $n = 7$, (0,0,1) is a solution because $4(0) + 5(0) + 7(1) = 7$ **(1 pt)**
$n = 8$, (2,0,0) is a solution because $4(2) + 5(0) + 7(0) = 8$ **(1 pt)**
$n = 9$, (1,1,0) is a solution because $4(1) + 5(1) + 7(0) = 9$ **(1 pt)**
$n = 10$, (0,2,0) is a solution because $4(0) + 5(2) + 7(0) = 10$ **(1 pt)**

Inductive Hypothesis: Assume for all integers, n, $7 \leq n \leq k$, where k is arbitrarily chosen and $k \geq 10$, that there exists a non-negative integer solution to $4x + 5y + 7z = n$. **(2 pts, give 1 if only assumed for k.)**

Inductive Step: Prove for $n = k+1$ that there is at least one non-negative integer solution to $4x + 5y + 7z = k+1$. **(1 pt)**

Via the inductive hypothesis, we know that there exists some solution (a, b, c) to the equation:

$4a+5b+7c = k-3$ **(2 pts to use this)**

This is because $k \geq 10$, so $k - 3 \geq 7$, and the inductive hypothesis applies for this quantity.

Now, consider the ordered triplet (a+1, b, c):

$$
\begin{aligned}
4(a + 1) + 5b + 7c &= 4a + 5b + 7c + 4 \\
&= (k - 3) + 4, \text{ using the inductive hypothesis} \\
&= k + 1
\end{aligned}
$$
(1 pt to add 1 to a/x)

It follows that a non-negative integer solution to $4x + 5y + 7z = k+1$ is (a+1, b, c). This solution is non-negative because $a, b, c \geq 0$, so it follows that $a+1 \geq 0$ as well.

Thus, we've proved the inductive step. It follows that for all integers $n \geq 7$, there exists at least one non-negative integer solution to $4x + 5y + 7z = n$.

Note: This problem IS the chicken nugget problem where you can buy 4 packs, 5 packs and 7 packs, x represents the number of 4 packs bought, y represents the number of 5 packs bought, and z represents the number of 7 packs bought.

8) (10 pts) Define a sequence of integers, $s_0, s_1, s_2, \ldots$ as follows:

$$s_0 = 3,\ s_1 = 25, \text{ for all integers } n \geq 2,\ s_n = 10s_{n-1} - 25s_{n-2}$$

Prove, using strong induction on n, for all non-negative integers, n, that $s_n = (2n+3)5^n$. (Note: You will use 2 base cases.)

Base cases: $n = 0$, LHS = $s_0 = 3$, RHS = $(2(0) + 3)5^0 = 3(1) = 3$ **(1 pt)**
$n = 1$, LHS = $s_1 = 25$, RHS = $(2(1) + 3)5^1 = 5(5) = 25$ **(1 pt)**

Thus, the assertion is true for both n = 0 and n = 1.

Inductive Hypothesis: Assume for all non-negative integers n, where $n \leq k$, where k is an arbitrarily chosen positive integer, that $s_n = (2n+3)5^n$. **(2 pts, 1 pt if only for k)**

Inductive Step: Prove for n = k+1 that $s_{k+1} = (2(k+1)+3)5^{k+1} = (2k + 5)5^{k+1}$ **(1 pt)**

$s_{k+1} = 10s_k - 25s_{k-1}$, via the definition of s. **(1 pt)**

$= 10(2k+3)5^k - 25(2(k-1)+3)5^{k-1}$, using the IH for both k and k-1. **(2 pts)**

$= 2(2k + 3)5^{k+1} - (2k - 2 + 3)5^{k+1}$

$= 5^{k+1}[2(2k+3) - (2k + 1)]$

$= 5^{k+1}[4k + 6 - 2k - 1]$

$= (2k + 5)5^{k+1}$ **(Rest of the Algebra: 2 pts)**

This completes the proof of the inductive step. It follows that for all non-negative integers, n, that $s_n = (2n + 3)5^n$.

9) (3 pts) With which famous scientist does the restaurant Einstein Brothers Bagels share a name?

Einstein (give 3 pts to all who submitted exam)