COT 3100 Sections 2/201 Exam #1 Solutions (9/15/2022)

1) (6 pts) Fill out the following truth table. Please place a T or F in each empty slot. Any ambiguous letter (according to the grader) will be marked incorrect.

p	q	r	$p \vee q$	$q \vee \bar{r}$	$(p \vee q) \wedge (q \vee \bar{r})$
F	F	F	F	T	F
F	F	T	F	F	F
F	T	F	T	T	T
F	T	T	T	T	T
T	F	F	T	T	T
T	F	T	T	F	F
T	T	F	T	T	T
T	T	T	T	T	T

Grading: 2 pts for each column, 1 pt if 5 to 7 entries correct, 2 pts if fully correct.

2) (10 pts) Using the laws of logic show that the two following logical expressions are equivalent:

(1) $(p \vee (\bar{p} \wedge q)) \vee (r \wedge (\bar{q} \vee r))$ (2) $p \vee q \vee r$

For the purposes of grading, please do NOT skip steps such as Commutative. (There are 2 of these.) There should be 6 steps total, so 7 lines written for your solution, at a minimum.

Step	Reason
$(p \vee (\bar{p} \wedge q)) \vee (r \wedge (\bar{q} \vee r))$	Given
$(p \vee (\bar{p} \wedge q)) \vee (r \wedge (r \vee \bar{q}))$	Commutative Law
$(p \vee (\bar{p} \wedge q)) \vee r$	Absorption Law
$((p \vee \bar{p}) \wedge (p \vee q)) \vee r$	Distributive Law
$(T \wedge (p \vee q)) \vee r$	Inverse Law
$((p \vee q) \wedge T) \vee r$	Commutative Law
$p \vee q \vee r$	Identity Law

Grading: 1 pt off for missing step, 1 pt off for missing or incorrect reason, cap at 10 pts off.

3) (5 pts) Prove the following argument via the Rules of Inference. You may skip the Commutative Step in this problem. (It comes up once.)

$$p \rightarrow q$$
$$r \rightarrow s$$
$$p \vee r$$
$$\bar{s}$$

$$\therefore q$$

Number	Step	Reason
1	$p \rightarrow q$	Given
2	$r \rightarrow s$	Given
3	$p \vee r$	Given
4	$q \vee s$	Constructive Dilemma w 1,2,3
5	$\bar{s}$	Given
6	q	Disjuctive Syllogism
7		
8		

Grading: 1 pt for all given steps (listing them), 2 pts for each of the other 2 steps, 1 pt for the step, 1 pt for the reason.

4) (4 pts) Prove or disprove the following assertion over the universe of real numbers:

$$\exists x \forall y [xy = 1]$$

Please clearly note whether the assertion is true or not, followed a justification of your answer. Most of the points are awarded for the justification.

This is false. There can be no value of x such that for all different values of y, xy = 1. To prove this, note that if we choose y = 0, then no matter what x is, $xy = x(0) = 0 \neq 1$.

Grading: 2 pts for stating false (0 if they said it was true for the whole question.)
2 pts for reason

5) (10 pts) Prove or disprove the following assertion for all sets A, B, C and D. **If you prove, please use direct proof or proof by contradiction, do NOT use a set membership table.**

$$(A \cup B) \times (C \cup D) \subseteq (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$$

This is true. We will prove it via direct proof (split into 4 cases):

Let $(x, y) \in (A \cup B) \times (C \cup D)$, be an arbitrarily chosen element. We aim to prove that $(x, y) \subseteq (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$, under this assumption.

By definition of Cartesian Product, we must have that

$x \in (A \cup B) \wedge y \in (C \cup D)$

By definition of union, we have the following:

$((x \in A) \vee (x \in B)) \wedge ((y \in C) \vee (y \in D))$

Using the Distributive Law (twice), this gives us the following four options, each separated by an or:

$((x \in A) \wedge (y \in C)) \vee ((x \in A) \wedge (y \in D)) \vee ((x \in B) \wedge (y \in C)) \vee ((x \in B) \wedge (y \in D))$

In the four cases, by definition of Set Product, we have the following:

$(x, y) \in A \times C \vee (x, y) \in A \times D \vee (x, y) \in B \times C \vee (x, y) \in B \times D$

Since in all four cases we have the membership of (x, y) in one of the four designated sets, by definition of set union, we can conclude that

$(x, y) \subseteq (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$, as desired.

Grading: Note, student solution can be more concise and take shorter more intuitive steps and still get full credit.

+2 for stating it's true (0/10 automatic for any disproof)
+2 for $(\boldsymbol{x}, \boldsymbol{y}) \in (\boldsymbol{A} \cup \boldsymbol{B}) \times (\boldsymbol{C} \cup \boldsymbol{D})$
+1 for $\boldsymbol{x} \in (\boldsymbol{A} \cup \boldsymbol{B}) \wedge \boldsymbol{y} \in (\boldsymbol{C} \cup \boldsymbol{D})$
+1 for $((\boldsymbol{x} \in \boldsymbol{A}) \vee (\boldsymbol{x} \in \boldsymbol{B})) \wedge ((\boldsymbol{y} \in \boldsymbol{C}) \vee (\boldsymbol{y} \in \boldsymbol{D}))$
+4 for all four cases worked out.

6) (10 pts) Prove or disprove the following assertion for all sets A, B and C. **If you prove, please use direct proof or proof by contradiction, do NOT use a set membership table.**

$$if\ A \subseteq B \cap \bar{C}, then\ (B - A) \cap C = \emptyset$$

This is false. Consider the following counter-example:

$A = \emptyset$, $B = \{1\}$, $C = \{1\}$

In this case, the if statement is true because the empty set is a subset of all sets. But, in this case we have:

$B - A = \{1\}$, $C = \{1\}$, and $(B - A) \cap C = \{1\} \neq \emptyset$. Thus, the assertion isn't true in all cases.

Grading: 0/10 if stated true.
+3 for saying false
+2 for any attempt at a counter-example
+4 for that counter-example actually being a counter-example to the claim
+1 for explaining why the counter-example is a counter-example

7) (5 pts) Let A, B and C be sets such that $|A| = 20, |B| = 30, |C| = 40$ and $|A \cap B| = 10$. What is the maximum possible value of $|A \cup B \cup C|$?

By the Inclusion-Exclusion Principle, we have

$$|A \cup B| = |A| + |B| - |A \cap B| = 20 + 30 - 10 = 40$$

Now, apply the Inclusion-Exclusion Principle to the sets $(A \cup B)$ and C:

$$|(A \cup B) \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C| = 40 + 40 - |(A \cup B) \cap C|$$

If we aim to maximize this value, we aim to minimize $|(A \cup B) \cap C|$. Based on the given information, there's no reason this value can't be 0, since no restriction is given on C. Thus, the desired maximum value of $|A \cup B \cup C|$ is 80.

Grading: 1/5 for answers of 90 or any use of I/E that doesn't make real progress.
Max 3/5 for answer of 70 or 80 – stuff (variables), (range 1-3)
0/5 if > 90 unless error is clearly only arithmetic (and then 1/5)
80 with reasonable work is worth 5/5, 80 out of the blue is worth 3/5

Note: For Honors class, +3 for answer of 80, +2 for reason

8) (8 pts) Janna does a hike up and down a mountain. She travels at a rate of 2 miles per hour going up the mountain and 4 miles per hour coming down. The total time of her hike is 7 hours (she's a pretty serious hiker =)) and the distance she traversed going down the mountain is 50% greater than the distance going up the mountain. How long was her full hike up and down the mountain?

Let D miles be the distance up the mountain. Then $\frac{3}{2}D$ is the distance down the mountain. (The total trip distance is $D + \frac{3}{2}D = \frac{5}{2}D$.) Since we know the rate of travel for both portions of the trip, we can calculate the time, in hours for going up the mountain and going down the mountain:

$$t_{up} = \frac{D}{2}, \qquad t_{down} = \frac{\frac{3}{2}D}{4} = \frac{3D}{8}$$

We are given that these two times sum to 7 hours, so set up the equation:

$$\frac{D}{2} + \frac{3D}{8} = 7$$

$$\frac{4D + 3D}{8} = 7$$

$$\frac{7D}{8} = 7$$
$$D = 8$$

It follows that the total trip was $\frac{5}{2} \times 8 = 20$ miles long.

Grading: 1 pt creating a distance variable of some sort, 2 pts for time up in terms of dist var, 2 pts for time down in terms of dist var, 1 pt for setting sum equal to 7, 2 pts to get to final answer. (Just 1 pt off if answer with wrong variable but did all the work.)

9) (6 pts) Logarithms were originally invented to help astronomers make multiplication calculations easier. Let a = 2.2 x 10^8 by b = 3.1 x 10^9. If $log_{10}2.2 = .3424$ and $log_{10}3.1 = .4914$, what is the value of $log_{10}ab$? (Note: An anti-log table would be used in conjunction with the answer to this question to determine the product ab. Also, these tables were more precise than indicated in this question. The goal of the question is simply to illustrate the original use of logarithms!)

$$log_{10}ab = log_{10}a + log_{10}b = log_{10}(2.2 \times 10^8) + log_{10}(3.1 \times 10^9)$$

$$= log_{10}2.2 + log_{10}10^8 + log_{10}3.1 + log_{10}10^9$$

$$= .3424 + 8 + .4914 + 9$$
$$= 17.8338$$

Grading: 2 pts for first step, 2 pts to split into 4 log terms, 2 pts for final answer.

10) (10 pts) Find the value of x which satisfies the following equation:

$$log_9(3x^5) + log_{27}(81x^4) = 21$$

$log_9(3x^5) + log_{27}(81x^4) = 21$

$\frac{log_3(3x^5)}{log_3 9} + \frac{log_3(81x^4)}{log_3 27} = 21$

$\frac{log_3 3 + log_3 x^5}{2} + \frac{log_3 81 + log_3 x^4}{3} = 21$

$\frac{1 + 5log_3 x}{2} + \frac{4 + 4log_3 x}{3} = 21$

Let $A = log_3 x$, and multiply through by 6:

$3(1 + 5A) + 2(4 + 4A) = 126$

$3 + 15A + 8 + 8A = 126$

$23A = 115$

$A = 5$, thus $5 = log_3 x$ and $x = 3^5 = \mathbf{243}$

Grading: 2 pts change base, 2 pts split numerators, 2 pts eval constants, 3 pts to get what $\boldsymbol{log_3 x}$ is, 1 pt to get answer for x.

11) (1 pt) Dan Marino's (a former NFL quarterback) 61st birthday is today. After which NFL quarterback is your instructor's email address named after?

Dan Marino (**Grading: give to all)**