Final Exam Review

Sunday, November 29, 2020 1:02 PM

1) Table of how far percy has gone and how much time he has spent:

Mile	Time	Avg MPH
1	1	1
1	1/2	2/1.5 = 1.3333
1	1/3	3/(11/6) = 1.6364
1	1/4	4/(25/12) = 1.92
1	1/5	5/(137/60) = 2.1898

Each mile, we complete it faster. Specifically, the kth mile takes us 1/k hours. So, we can just build a chart, mile by mile that shows us our total average speed of the trip by adding up the time it took us to get to each mile marker.

k = 5 is the first time our average speed exceeds 2 miles per hour. At this point our average speed as a fraction is 300/137 miles per hour.

2) $f(2) = f(1)^4 = 2^4 = 16$
$f(3) = f(2)^8 = 16^8 = (2^4)^8 = 2^{32}$
$f(4) = f(3)^{16} = 2^{32 \times 16} = (2^{16})^{32}$

$\log_{65536}(2^{16})^{32} = 32$, by definition of log

$$\log_{65536} 32 = \frac{\log_2 32}{\log_2 65536} = \boxed{\frac{5}{16}}$$

6) Base case is n = a.

$$LHS = \sum_{i=a}^{a} \binom{i}{a} = \binom{a}{a} = 1 \checkmark$$

$$RHS = \binom{a+1}{[illegible]} = 1 \checkmark$$

$$RHS = \binom{a+1}{a+1} = 1 \checkmark$$

Base case is true for n = a.

Inductive Hypothesis: Assume for an arbitrarily chosen integer n = k, where k >= a that

$$\sum_{i=a}^{k} \binom{i}{a} = \binom{k+1}{a+1}$$

Inductive Step: Prove for n = k+1 that

$$\sum_{i=a}^{k+1} \binom{i}{a} = \binom{k+2}{a+1}$$

$$\sum_{i=a}^{k+1} \binom{i}{a} = \sum_{i=a}^{k} \binom{i}{a} + \binom{k+1}{a}$$

$$= \binom{k+1}{a+1} + \binom{k+1}{a}, \text{ using IH}$$

$$= \binom{k+2}{a+1} \text{ using Pascal}.$$

$$= \binom{k+2}{a+1} \quad \text{Pascal Triangle Identity}$$

This proves the inductive step. Thus, we can conclude the claim is true for all integers n >= a.

7a)2As, 2Ns, Q, U, R, T, I, E
Just use regular permutation formula noting there are 2As and 2Ns

$$\frac{10!}{2!2!}$$

7b) One approach: take regular ones and sub out ones either AA or NN, but when you over count, you have to sub out ones with both AA and NN.

Perms with AA: treat AA as a super letter, so there are 9 letters total, and the only double letter is N,

$$\frac{9!}{2!}$$

Perms with NN: treat NN as a super letter, so there are 9 letters total, and the only double letter is A

$$\frac{9!}{2!}$$

Perms with both AA and NN: Treat both as super letters and now we have 8 unique letters which we can arrange in 8! ways.

Final answer: $\boxed{\frac{10!}{2!2!} - 2\left(\frac{9!}{2!}\right) + 8!}$

$$\frac{10!}{4} - 9! + 8!$$

$$= 8!\left(\frac{45}{2} - 9 + 1\right)$$

$$= \boxed{8! \cdot \frac{29}{2}}$$

Another way that is pretty tricky is to use all the other letters as separators.

6 unique letters… insert the As 6! for the other letters and then the As can be selected in 7 choose 2 ways.

_QAU_R_T_IAE _

$6!\binom{7}{2}$

Insert N's in $\binom{9}{2}$ ways

ANA is not counted above

This gets stuck because we either forget to count some stuff or accidentally count some stuff too much. But I do think there is a way to get this to work.

7c) U has to be at the top:
As on both sides of U
Ns on both sides of U
5 letters left to place.
Each can be placed on either side of the U

A NQRUTNIEA

fix

Ns on both sides of U
5 letters left to place.
Each can be placed on either side of the U,
so two choices for each letter, this can be done
in $2^5 = 32$ ways.

Q, T, I, E, R

once we fix direction, permutation is fixed.

9) reflexive: yes (a, a) is in R because $|a - a| = 0 < 5$.
irreflexive: no because (1, 1) is in R, proof above works to show that.
symmetric: yes, if (a,b) in R then (b,a) in R.
Let (a,b) be an arbitrary element in R
This means that $| a - b | < 5$.
$|a - b| = |b - a| < 5$, proving that (b, a) is in R
antisymmetric: no (1,2) is in R (2, 1) is in R and 1 is not equal 2.
transitive: no, (1, 4) is in R and (4, 7) is in R but (1, 7) is not in R.

10) $P(x) = (x-4)(x-5)(x-6)(x-7)(x-8) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$
$P(-1) = (-5)(-6)(-7)(-8)(-9) = -1 + a - b + c - d + e$
$-5x6x7x8x9 + 1 = a - b + c - d + e$
The value of a -b +c-d+e = $1 - \frac{9!}{4!}$