

Final Exam Review

Sunday, November 29, 2020 1:02 PM

- 1) Table of how far Percy has gone and how much time he has spent:

Mile	Time	Avg MPH
1	1	1
1	1/2	2/1.5 = 1.3333
1	1/3	3/(11/6) = 1.6364
1	1/4	4/(25/12) = 1.92
1	1/5	5/(137/60) = 2.1898

Each mile, we complete it faster. Specifically, the k th mile takes us $1/k$ hours. So, we can just build a chart, mile by mile that shows us our total average speed of the trip by adding up the time it took us to get to each mile marker.

$k = 5$ is the first time our average speed exceeds 2 miles per hour. At this point our average speed as a fraction is $300/137$ miles per hour.

$$\begin{aligned} 2) f(2) &= f(1)^4 = 2^4 = 16 \\ f(3) &= f(2)^8 = 16^8 = (2^4)^8 = 2^{32} \\ f(4) &= f(3)^{16} = 2^{32 \times 16} = (2^{16})^{32} \end{aligned}$$

$\log_{65536}(2^{16})^{32} = 32$, by definition of log

$$\log_{65536} 32 = \frac{\log_2 32}{\log_2 65536} = \boxed{\frac{5}{16}}$$

- 6) Base case is $n = a$.

$$\begin{aligned} \text{LHS} &= \sum_{j=a}^a \binom{j}{a} = \binom{a}{a} = 1 \quad \checkmark \\ \dots & \binom{a+1}{a} = 1 \quad \checkmark \end{aligned}$$

$$RHS = \binom{a+1}{a+1} = 1 \quad \checkmark$$

Base case is true for $n = a$.

Inductive Hypothesis: Assume for an arbitrarily chosen integer $n = k$, where $k \geq a$ that

$$\sum_{j=a}^k \binom{j}{a} = \binom{k+1}{a+1}$$

Inductive Step: Prove for $n = k+1$ that

$$\sum_{i=a}^{k+1} \binom{i}{a} = \binom{k+2}{a+1}$$

$$\sum_{i=a}^{k+1} \binom{i}{a} = \sum_{i=a}^k \binom{i}{a} + \binom{k+1}{a}$$

$$= \binom{k+1}{a+1} + \binom{k+1}{a} \quad \text{using Pascal's}$$

$$= \binom{k+2}{a+1} \quad \text{using Pascal's}$$

$$= \binom{k+l}{a+1}$$

Pascal's Triangle Identity ✓

This proves the inductive step. Thus, we can conclude the claim is true for all integers $n \geq a$.

7a) 2As, 2Ns, Q, U, R, T, I, E
Just use regular permutation formula noting there are 2As and 2Ns

$$\frac{10!}{2!2!}$$

7b) One approach: take regular ones and sub out ones either AA or NN, but when you over count, you have to sub out ones with both AA and NN.

Perms with AA: treat AA as a super letter, so there are 9 letters total, and the only double letter is N,

$$\frac{9!}{2!}$$

Perms with NN: treat NN as a super letter, so there are 9 letters total, and the only double letter is A

$$\frac{9!}{2!}$$

Perms with both AA and NN: Treat both as super letters and now we have 8 unique letters which we can arrange in $8!$ ways.

Final answer:

$$\frac{10!}{2!2!} - 2 \left(\frac{9!}{2!} \right) + 8!$$

$$\frac{10!}{4} - 9! + 8!$$

$$= 8! \left(\frac{45}{2} - 9 + 1 \right)$$

$$= 8! \cdot \frac{29}{2}$$

Another way that is pretty tricky is to use all the other letters as separators.

6 unique letters... insert the As 6! for the other letters and then the As can be selected in 7 choose 2 ways.

Q A U R T I A E

$$6! \binom{7}{2}$$

Insert N's in $\binom{9}{2}$ ways

ANA is not counted above

This gets stuck because we either forget to count some stuff or accidentally count some stuff too much. But I do think there is a way to get this to work.

7c) U has to be at the top:

As on both sides of U

Ns on both sides of U

5 letters left to place.

Each can be placed on either side of the U

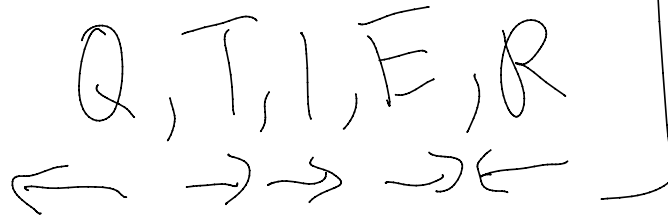
A N Q R U T N I E A

fix

Ns on both sides of U

5 letters left to place.

Each can be placed on either side of the U,
so two choices for each letter, this can be done
in $2^5 = 32$ ways.



once we fix
direction,
permutation is
fixed.

9) reflexive: yes (a, a) is in R because $|a - a| = 0 < 5$.

irreflexive: no because $(1, 1)$ is in R, proof above works to show that.

symmetric: yes, if (a, b) in R then (b, a) in R.

Let (a, b) be an arbitrary element in R

This means that $|a - b| < 5$.

$|a - b| = |b - a| < 5$, proving that (b, a) is in R

antisymmetric: no $(1, 2)$ is in R $(2, 1)$ is in R and 1 is not equal 2.

transitive: no, $(1, 4)$ is in R and $(4, 7)$ is in R but $(1, 7)$ is not in R.

10) $P(x) = (x-4)(x-5)(x-6)(x-7)(x-8) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$

$P(-1) = (-5)(-6)(-7)(-8)(-9) = -1 + a - b + c - d + e$

$-5 \times 6 \times 7 \times 8 \times 9 + 1 = a - b + c - d + e$

The value of $a - b + c - d + e =$

$$1 - \frac{9!}{4!} = 2$$