COT 3100 Recitation #13: Probability 2 Solutions
11/16/2020-11/20/2020

Problem Solved in Recording

1. The numbers 1, 2, 3, …, 9 are randomly placed into the 9 squares of a 3 x 3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

Solution

Sample Space…There are 9 slots, we have 5 odd numbers, we must choose 5 out of 9 slots to place those odd numbers. So our sample space is $\binom{9}{5}$.

Key Observation, each row must have an odd number of odd numbers. (Same for columns.) From 1 to 9, we have 5 odd integers. We have three rows. The only way that 5 odd numbers can be distributed in 3 rows, so that each row has an odd number of odd numbers is (1, 1, 3), (1, 3, 1) or (3, 1, 1) designation. So, let's take an arbitrary look at one of these choices just for the rows:

X	X	X
		X
		X

Once I choose the column for three, then I have to choose the row for three:

	X	
	X	
	X	

So after choosing column 2 to store three odd numbers, let's say I choose row 1 to store three odds:

X	X	X
	X	
	X	

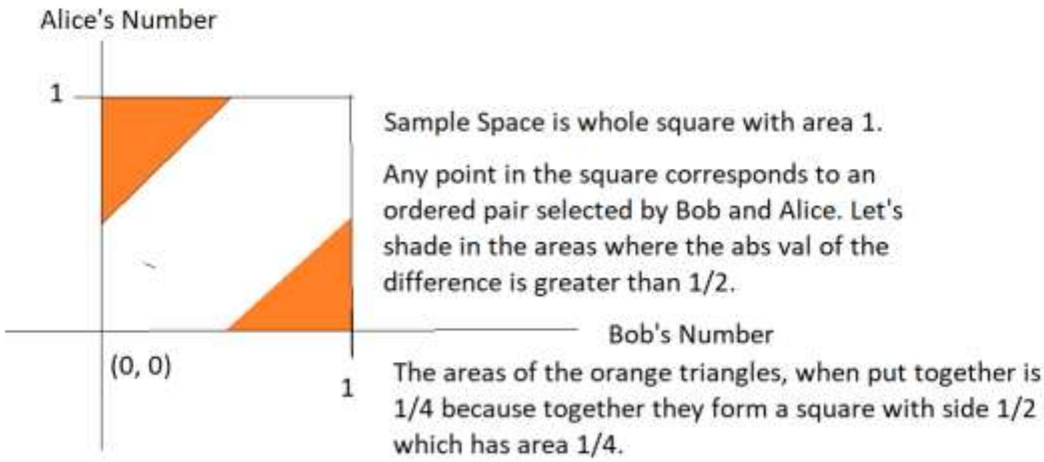
The number of ways we can satisfy the prompt is # of columns times the # of rows. So we can get an odd sum for all rows and columns in 3 x 3 ways.

Desired probability = $\frac{9}{\binom{9}{5}} = \frac{9(5!4!)}{9!} = \frac{5!4!}{8!} = \frac{4!}{8(7)(6)} = \frac{24}{7(48)} = \mathbf{\frac{1}{14}}$

2. Alice and Bob each choose a real number in between 0 and 1, with uniform distribution. What is the probability that the absolute value of the difference between their numbers exceeds .5?

Solution

We can visualize our sample space as a 2D grid of the number Alice chooses on one axis and the number Bob chooses on the other. The solution is simply taking the area of the sections of the square where the situation is satisfied and dividing that by the area of the square:



3. Let N be a positive multiple of 5. One red ball and N green balls are arranged in a line in random order. Let P(N) be the probability that at least 60% of the green balls are on the same side of the red ball. Observe that P(5) = 1 and that P(N) approaches 80% as N grows large. What is the minimum value of N for which P(N) < $\frac{321}{400}$?

Solution

Let N = 5k, k is positive integer. Let's find P(N) = P(5k).

5k + 1 balls, one is red. Let the position values be 0 to 5k, inclusive. If we place the red ball in positions 0 through 2k, inclusive, there are 3k or more green balls to the right. If we place the red ball in positions 3k through 5k, inclusive, there are 3k or more green balls to the left.

Out of the 5k+1 places I could place the red ball, I get a success in any of the 2k+1 + 2k + 1 = 4k+2 locations previously mentioned.

$$\frac{4k+2}{5k+1} < \frac{321}{400}$$

$$400(4k+2) < (5k+1)(321)$$
$$1600k + 800 < 1605k + 321$$
$$479 < 5k$$
$$N > 479$$

Since N is a multiple of 5, N = **480** is the smallest such value.

4. A coin is altered so that the probability that it lands on heads is less than $\frac{1}{2}$ and when the coin is flipped four times, the probability of an equal number of heads and tails is $\frac{1}{6}$. What is the probability that the coin lands on heads?

Solution

Let p be the probability of landing heads. The probability we get exactly 2 heads is:

$$\binom{4}{2} p^2 (1-p)^2 = \frac{1}{6}$$
$$6p^2(1-p)^2 = \frac{1}{6}$$
$$p^2(1-p)^2 = \frac{1}{36}$$

Let X = p(1-p), so we will solve for X first.

$$X^2 = \frac{1}{36}$$

Since X > 0, X = $\frac{1}{6}$. Now, go back to the substitution, so we get:

$$p(1-p) = \frac{1}{6}$$
$$6p(1-p) = 1$$
$$6p^2 - 6p + 1 = 0$$

$$p = \frac{6 \pm \sqrt{36 - 4(6)}}{12} = \frac{6}{12} \pm \frac{\sqrt{12}}{12} = \frac{3 \pm \sqrt{3}}{6}$$

Since p < ½, we have $p = \frac{3-\sqrt{3}}{6}$.

Problems for Recitation

1. If the number is selected at random from the set of all five-digit numbers in which the sum of the digits is equal to 43, what is the probability that the number will be divisible by 11? (fall 18 final rev)

Solution

Let the number be abcde, with a + b + c + d + e = 43 and each are digits. Notice that 5 x 9 = 45, so we must either have four 9s and one 7 or three 9s and 2 8s. Thus, our sample space is $\frac{5!}{4!1!}+\frac{5!}{3!2!}=5+10=15$, since we are permuting either four 9s and one 7 or three 9s and two 8s. (The first of the terms above counts the permutations of 9s and 7s while the second term counts the permutations of 9s and 8s.) Now, of these 15 five digit numbers, we must count how many are divisible by 11.

Now, recall the divisibility rule for 11. Take the alternative digit sum and it must be equivalent to 0 mod 11. Put another way, a + c + e ≡ b + d (mod 11). For 9s and 8s, we see that the only solution is 9 + 9 + 9 ≡ 8 + 8 (mod 11). (It's easy to see that swapping any 8 or 9 will ruin the equivalence.) This equivalence maps to only one five digit number: 98989.

For 9s and 7s, we see that 9 + 9 + 9 ≡ 9 + 7 (mod 11). This equivalence maps to two five digit numbers: 99979 and 97999.

It follows that the desired probability is just $\frac{3}{15}=\mathbf{\frac{1}{5}}$.

2. Alex, Mel, and Chelsea play a game that has 6 rounds. In each round there is a single winner, and the outcomes of the rounds are independent. For each round the probability that Alex wins is 50%, and Mel is twice as likely to win as Chelsea. What is the probability that Alex wins three rounds, Mel wins two rounds, and Chelsea wins one round? (Prob 11 2012 AMC-12A)

Solution

This is a binomial distribution with 3 choices instead of 2. We can choose Alex's three winning rounds in $\binom{6}{3}$ ways. We can choose Mel's remaining 2 wins out of 3 remaining rounds in $\binom{3}{2}$, thus there are $\binom{6}{3}\binom{3}{2}=60$ orderings of outcomes with Alex winning 3, Mel winning 2 and Chelsea winning one round. Alex's probability of winning is 1/2, and the sum of Mel and Chelsea's probabilities of winning is also 1/2. But since Mel's probability of winning is twice of Chelsea, his probability of winning is 1/3 and hers is 1/6. (You can set up equations to formally see this, but these ratios are pretty common in probability problems with 3 options, so it is easy to see that these are the desired probabilities.) Each of these sequences of results occurs with probability $\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(\frac{1}{6}\right)=\frac{1}{8\times9\times6}$. It follows that the desired probability is $\frac{60}{8\times9\times6}=\frac{10}{72}=\mathbf{\frac{5}{36}}$.

3. A pair of standard 6-sided dice is rolled once. The sum of the numbers rolled determines the diameter of a circle. What is the probability that the numerical value of the area of the circle is less than the numerical value of the circle's circumference?

Solution

The circumference of a circle with radius r is $2\pi r$, and the area of the same circle is πr^2. Solve for the area less than the circumference:

$$\pi r^2 < 2\pi r$$
$$r < 2$$

This means that d, the diameter of the circle is less than 4. Thus, the diameter must be either 2 or 3. We roll a 2 with probability 1/36 and a 3 with probability 2/36. The probability we roll 2 or 3 then is 3/36 = **1/12.**

4. Ten children are standing in line. Each child flips a fair coin. For each child whose coin lands heads, they get a candy. What is the probability that no two children who are adjacent in line both receive candy?

Solution

There are 2^{10} strings of 10 letters, all of which are either T or H, for flipping heads or tails. This is our sample space. Of these, we must count how many have no two adjacent heads.

Let t(n) = # of strings of length n with no two heads adjacent.

t(1) = 2, for H or T
t(2) = 3, for TT, TH and HT.

Now, consider trying to form a string of n+1 letters with no consecutive H's.

We can take all strings of length n and prepend a T to them. There are t(n) ways to do this.
We can take all strings of length n-1 and prepend a HT to them. There are t(n-1) ways to do this. This count misses no strings and all strings in the first batch start with T and all strings in the second batch start with H, so there is no overlap in our counting. Thus, we can add. This gives us the recurrence:

t(n+1) = t(n) + t(n-1), the same recurrence as the Fibonacci sequence.

We can calculate t(10) iteratively:

t(3) = 5, t(4) = 8, t(5) = 13, t(6) = 21, t(7) = 34, t(8) = 55, t(9) = 89 and t(10) = 144.

The desired probability is $\frac{144}{1024} = \mathbf{\frac{9}{64}}$.