COT 3100 Recitation #11: Counting 2
Quiz and Solutions

11/2/2020-11/6/2020

1) How many integers in between 1 and 1000 are divisible by 7, 11 or 13?

Solution

We use the Inclusion-Exclusion Principle and the fact that there are $\left\lfloor \frac{n}{k} \right\rfloor$ integers in between 1 and n, inclusive, that are divisible by k. Let set A be the set of values in range divisible by 7, set B be the set of values in range divisible by 11 and set C be the set of values in range divisible by 13. Using the I/E Principle for 3 sets we get the following:

$$\left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor + \left\lfloor \frac{1000}{13} \right\rfloor - \left\lfloor \frac{1000}{77} \right\rfloor - \left\lfloor \frac{1000}{91} \right\rfloor - \left\lfloor \frac{1000}{143} \right\rfloor + \left\lfloor \frac{1000}{1001} \right\rfloor$$

$$= 142 + 90 + 76 - 12 - 10 - 6 + 0 = \mathbf{280}$$

2) A lattice point is a point on the Cartesian plane with integer coordinates for both x and y. How many lattice points are on the line segment connected by the points (13, 16) and (157, 124)? **Please include BOTH endpoints in your count.**

Solution

As discussed in the recitation video, the gcd(157 - 13, 124 - 16) represents the ***"number of hops"*** it would take jumping on all the lattice points to get from (13, 16) to (157, 124). If you make n jumps, then you are visiting n+1 locations. Thus, the final answer is 1 + gcd(157-13, 124 - 16). Now, let's calculate that gcd:

144 = 1 x 108 + **36**
108 = 3 x 36

The desired GCD is 36, thus there are a total of **37** lattice points on the line segment form (13, 16) to (157, 124), including both endpoints.

3) How many solutions to the equation a + b + c + d + e + f = 100 are there where each of the six variables, a, b, c, d, e and f are **positive even integers**?

Solution

Since each are even, let a = 2a', b = 2b', c = 2c', d = 2d', e = 2e' and f = 2f', where a', b', c', d', e' and f' are all **positive integers**. Then we want the number of solutions to:

2a' + 2b' + 2c' + 2d' + 2e' + 2f' = 100
a' + b' + c' + d' + e' + f' = 50

Since each of these must be positive, "**give 1 to each of the 6 variables**" in advance, so that we have 44 more units to distribute freely amongst the variables. More formally, let a" = a' - 1, where a" is a non-negative integer and make similar substitutions for the other variables. Now, we seek the number of non-negative integer solutions to:

$$a'' + b'' + c'' + d'' + e'' + f'' = 44$$

This is a combinations with repetition problem with n = 44 and r = 6. There are $\binom{44+6-1}{6-1}$ ways, simplified we get $\binom{49}{5}$ solutions to the given equation that satisfy all the restrictions.

4) How may non-negative integer solutions are there to the equation,

$$a + b + c + d + e + f + g \leq 76?$$

Solution

Add a dummy variable h to the equation to equal the difference from the sum of variables a to g. Thus, we seek the number of non-negative integer solutions to

$$a + b + c + d + e + f + g + h = 76$$

This is a combinations with repetition problem with n = 76 and r = 8. There are $\binom{76+8-1}{8-1}$ ways, simplified we get $\binom{83}{7} = \binom{83}{76}$ solutions. The latter was the correct answer choice listed.

5) A class of six students is seated in a grid of 2 rows and 3 columns. There are six students in the class: Aliyah, Benny, Cammy, Deshawn, Emily and Fran. Unfortunately, Cammy and Fran are trouble makers and can not sit next to each other. More specifically, the two of them can't sit beside, behind or diagonally from one another. In how many ways can the students be seated?

Solution

Label the seats 1, 2, 3 on the first row and 4, 5, 6 on the second row:

1	2	3
4	5	6

Let's brute force the placement of Cammy and Fran. Note that neither can sit in seats 2 and 5, as these are adjacent to all other seats.

Cammy - 1, Fran - either 3 or 6
Cammy - 2, Fran - either 3 or 6

Fran - 1, Cammy - either 3 or 6
Fran - 2, Cammy - either 3 or 6

Thus, there are 8 ways to seat Cammy and Fran. There are no restrictions on the remaining 4 students, so they can be placed in 4 x 3 x 2 x 1 = 24 ways, since we start off with 4 remaining open seats as we try to seat those 4 students.

It follows that the total number of valid seating charts is 8 x 24 = **192**.