Counting Principles

Addition: If I split up my counting into disjoint groups, then just add up each group.

13 girls, 14 boys how many people in the class? Answer = 13 + 14 = 27

$$if\ A \cap B = \emptyset, then, |A \cup B| = |A| + |B|$$

Example: Bob, Dave, Jim, Jack plus Sarah,Jennifer,Joslyn

So, 4 + 3 = 7 people in the class… we can visualize two separate groups and counting each item in each group…

Subtraction Principle: We want to count all the kids doing activities at P.E. 100 kids in P. E. total, but 3 kids got sick and did not do any activites, everyone else did some activities…

100 – 3 = 97 kids did activies.

Multiplication Principle, any time you are counting the # of ordered pairs (a, b) that satisfy some constraint and you have a fixed number choices for A and a fixed # of choices for b, just multiply those two numbers to get the total number of possible ordered pairs.

$$|A \times B| = |A||B|$$

Value meals: 1 entrée and 1 side, 5 choices of side, 7 choices of entrees, then we have a total possible # of meals = 5 x 7.

The table illustration is the easiest way to see it's multiplication.

	Burger	Taco	Chicken	Salad	Sandwich	Empanada	Steak
Fries	B+F	T+F	C+F	S+F	Sd+F	E+F	St+F
Nachos	B+N	T+N	C+N	S+N	Sd+N	E+N	St+N
Potato	B+P	T+P	C+P	S+P	Sd+P	E+P	St+P
Brownie	B+B	T+B	C+B	S+B	Sd+B	E+B	St+B
Churro	B+C	T+C	C+C	S+C	Sd+C	E+C	St+C

Division Principle: If I've counted each item I was supposed to count exactly k times, then divide my answer by k to get the real answer.

Combination: "n choose k" represents the number of ways to choose k items out of n.

N = 5, k = 3

M&m, snickers, Hershey bar, twix, reece's

M&m, snickers, Hershey
M&m, snickers, twix
M&m, snickers, reece's
M&m, Hershey, twix
M&m, Hershey, reeces
M&m, Twix, Reeces
snickers, Hershey bar, twix
snickers, Hershey bar, reece's
snickers, twix, reece's
Hershey bar, twix, reece's
So 5 choose 3 = 10

If I have n items, and I want to place k of those items in a line, how many ways can I do it?

10 items, and we want to place 4 in a line.

1^{st} item: 10 choices
2^{nd} item: 9 choices

A	(BCDEFGHIJ)
B	(ACDEFGHIJ)
C	(ABDEFGHIJ)
D	(ABCEFGHIJ)
E	(ABCDFGHIJ)
F	(ABCDEGHIJ)
G	(ABCDEFHIJ)
H	(ABCDEFGIJ)
I	(ABCDEFGHJ)
J	(ABCDEFGHI)

Here above is an illustration of the 90 ways we can choose the first 2 items in the line, illustrating the multiplication principle.

3^{rd} item: 8 choices
4^{th} item: 7 choices

10 x 9 x 8 x 7 = ${}_{10}P_4$ (# of ways to permute 4 items out of 10)

In terms of n and k, the answer is n(n-1)(n-2)…(n-k+1) = $\frac{n!}{(n-k)!}$.

$$\frac{10!}{6!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 10 \times 9 \times 8 \times 7$$

Now, imagine counting combinations. When we look at permutations, for example, the permutation ABCD was counted 24 times: ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, …., DCBA. The # of ways to permute ABCD is 4 x 3 x 2 x 1 = 4! This is exactly how many times, ABCD was overcounted above, if we only care to count combinations (where order doesn't matter…)

"n choose k" = $\binom{n}{k} = {}_nC_k = \frac{n!}{(n-k)!k!}$

Problems

1)How many three digit numbers are not divisible by 5, have digits that sum to less than 20, and have the first digit equal to the third digit?

1x1	6x6
2x2	7x7
3x3	8x8
4x4	9x9

If the first digit is in between 1 and 4 inclusive, then the middle digit can be anything because 4 + 4 + 9 < 20. **Number of numbers to count in this category is 4 x 10 = 40.**

6x6, x < 8, so x can be 0,1,2,3,…,7 (**8 choices)**
7x7, x < 6 so x can be 0,1,2,3…,5 **(6 choices)**
8x8, x < 4 so x can be 0,1,2,3 **(4 choices)**
9x9, x < 2 so x can be 0,1 **(2 choices)**

Final Answer = 4 x 10 + 8 + 6 + 4 + 2 = 60

2) Find the sum of the digits in all of the integers from 1 to 10000, inclusive.

Odometer spinning from

0000 to 9999, and then add 10000

For each digit location, each digit occurs an equal number of times. So in the units digit, each of 0 through 9 occur exactly 1000 times.

# of digits = 4

Sum of digits = $\sum_{i=0}^{9} i = \frac{9 \times 10}{2} = 45$

# of times each digit occurs = 1000

4 x 45 x 1000 + 1 = 180000 + 1 = **180,001**

3) One thousand unit cubes are put together into a 10 x 10 x 10 cube and the surface of this 10 x 10 x 10 cube is painted. Then, the unit cubes are pulled apart. How many of the 1000 unit cubes have at least one face painted?

It's easier to count the ones NOT painted!!!

The inner 8 x 8 x 8 = 512 cubes are NOT painted.

# of cubes with at least 1 face painted is 10 x 10 x 10 – 8 x 8 x 8 = 488.

Let's try to do it the hard way…

100 x 6 = 600 cubes on the faces

But, we counted some of these multiple times.

12 edges, each of length 8 (don't count corners) have 2 sides painted, so let's subtract out 12 x 8 = 96.

There are 8 corners, and each of these were counted 3 times in the original count. So let's subtract out 8 x 2.

600 - 96 – 16 = **488.**

This works also, but is more work and far more tricky to get correct, since you have to visualize the cube, really well.

4) How many different integers in between 100 and 999 have either three increasing digits or three decreasing digits?

136…an example…digits must be distinct…**once I choose the digits, the number is fixed.**

Say you pick 7, 4, 6…that corresponds to the number 467 only with digits in increasing order.

So there is a one to one correspondence between the number ways I can choose 3 digits out of 9 digits {1,2,3,4,5,6,7,8,9} and the number of increasing 3 digit numbers.

# of increase = $\binom{9}{3} = \frac{9\times8\times7}{1\times2\times3} = 84$

For decreasing, we are choosing 3 digits out of 10 {0,1,2,3,4,5,6,7,8,9}

# of decreasing = $\binom{10}{3} = \frac{10\times9\times8}{1\times2\times3} = 120$

Final answer = 84 + 120 = 204