

COT 3100 Recitation #6: Use of Factorization of Integers, Algebraic Expressions
9/28/2020-10/2/2020

Solutions

Problem Solved in Recording

1) The zeros of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a ?

Solution

The two roots of the equation are: $\frac{a \pm \sqrt{a^2 - 8a}}{2}$. If both of these are to be integers, then the difference between the two roots must be integral, which means that $\sqrt{a^2 - 8a}$ must be an integer, which means that $a^2 - 8a$ must be a perfect square. Without loss of generality, there exists a non-negative integer b such that:

$$\begin{aligned}a^2 - 8a &= b^2 \\a^2 - 8a + 16 - 16 &= b^2 \\(a - 4)^2 - 16 &= b^2 \\(a - 4)^2 - b^2 &= 16 \\(a - 4 + b)(a - 4 - b) &= 16\end{aligned}$$

Since $b \geq 0$, we have six possibilities:

$$\begin{aligned}a - 4 + b &= 4, a - 4 - b = 4 \\a - 4 + b &= -4, a - 4 - b = -4 \\a - 4 + b &= 8, a - 4 - b = 2 \\a - 4 + b &= -2, a - 4 - b = -8 \\a - 4 + b &= 16, a - 4 - b = 1 \\a - 4 + b &= -1, a - 4 - b = -16\end{aligned}$$

The first leads to the solution $b = 0, a = 8$. The second leads to the solution $b = 0, a = 0$. The third leads to the solution $b = 3, a = 9$. The fourth leads to the solution $b = 3, a = -1$. The last two lead to a non-integer solutions of a and b . Plugging back each of these values of a yields the following quadratics: $x^2 - 8a + 16 = 0, x^2 = 0, x^2 - 9x + 18 = 0$, and $x^2 + x - 2 = 0$. These have the roots $(4, 4), (0, 0), (3, 6)$, and $(1, -2)$, respectively, so all four are valid.

It follows that the sum of the possible values of a is $8 + 0 + 9 + -1 = 16$.

2) What is the smallest possible positive integer x such that $1260x = N^3$, where N is some positive integer?

Solution

$$1260x = N^3$$
$$2^2 3^2 5^1 7^1 x = N^3$$

N must have 2, 3, 5 and 7 in its prime factorization. At a minimum $N = 2 \times 3 \times 5 \times 7 = 210$. Substituting, we have:

$$2^2 3^2 5^1 7^1 x = 2^3 3^3 5^3 7^3$$
$$x = 2^1 3^1 5^2 7^2 = \underline{\underline{7350}}$$

3) The number $10!$, when written in base 12, ends in how many zeroes? (Challenge: figure out a fast algorithm to solve this problem when the factorial can be quite large and the base is anything that is easy to prime factorize.)

Solution

The brute force method:

$$10! = 3268800_{10}$$

Base conversion (until we hit a non-zero remainder):

$$3268800/12 = 302400 R0$$
$$302400/12 = 25200 R0$$
$$25200/12 = 2100 R0$$
$$2100/12 = 175 R0$$
$$175/12 = 14 R7, \text{ we can stop here.}$$

When converted to base 12, **10! ends in four zeroes.**

In general, we see that for a given prime p , p divides into $n!$ $\sum_{i=1}^n \left\lfloor \frac{n}{p^i} \right\rfloor$ times. To see this, note that if we divide p into n and take the floor, this tells us the number of times we "cancel" p for each multiple of p in the written product of $n!$. But, in doing so, we DIDN'T cancel 2 copies of p from the values $p^2, 2p^2$, etc. So, we want to divide by p^2 to get the second copies of the prime p from those terms. We continue in this fashion. We can stop the sum well before n - namely, we can stop as soon as i is such that $p^i > n$.

Now, to apply it to this problem, $12 = 2^2 3$. Thus, we can calculate the number of times 2 divides into $10!$ using the formula above and do the same for 3. Then we see which of the two items is the limiting factor. For example, 2 divides into $10!$ $5 + 2 + 1 = 8$ times, so 2^2 divides into $12!$ 4 times. 3 divides into $10!$ $3 + 1 = 4$ times. Thus, both primes limit the number of times 12 divides evenly into $10!$ and thus there are 4 factors of 12 in $10!$, meaning that in base 12, $10!$ ends in **4 zeroes.**

4) Let A , M and C be non-negative integers such that $A + M + C = 12$. What is the maximum value of $AMC + AM + MC + AC$?

Solution

This is a creative step. When you see the terms listed, you see all combinations of size 2 and size 3 of the given terms. One way to list out all combinations added together is multiplying each term plus one:

$$\begin{aligned}(A + 1)(M + 1)(C + 1) &= AMC + AM + MC + AC + A + M + C + 1 \\ &= (AMC + AM + MC + AC) + 12 + 1\end{aligned}$$

Thus, we have

$$(AMC + AM + MC + AC) = (A + 1)(M + 1)(C + 1) - 13$$

Our goal then, is to maximize the product $(A + 1)(M + 1)(C + 1)$. More clearly, let $A' = A + 1$, $M' = M + 1$ and $C' = C + 1$. Then our goal is to maximize $A'M'C'$ with the constraint that $A' + M' + C' = 15$. The generalization of the arithmetic-geometric mean states that the geometric mean of a set of positive numbers is always less than or equal to their arithmetic mean. In this case, the arithmetic mean of A' , M' and C' is 5. Thus, $\sqrt[3]{A'M'C'} \leq 5$. Alternatively, we have $A'M'C' \leq 5^3 = 125$. Equality is achieved (and this can be seen pretty easily) when each term is equal. Thus, by setting $A' = 5$, $M' = 5$ and $C' = 5$, we achieve a maximum of 125 for $A'M'C'$. It follows that the maximum value for the quantity in question is $125 - 13 = \mathbf{112}$.

Problems for Recitation

1) Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. How many possible values for k are there?

Solution

In general, the sum of the roots of a quadratic equation with a leading coefficient of 1 is the negative of the x coefficient. To see this, note that any quadratic of this form can also be written as $(x - r)(x - s)$, where r and s are the two roots of the equation. Multiplying this out we get:

$$x^2 - rx - sx + rs = x^2 - (r + s)x + rs$$

From here, if we equate this to the regular quadratic form of the equation, we see that the sum of the roots ($r+s$) is the opposite of the x coefficient.

Thus, the sum of the two roots of the equation is 63. One possible solution is that the two prime numbers in question are 2 and 61, respectively. There are no other possibilities for the roots because 2 is the only even prime number and if the two roots sum to be odd, precisely one must be even. **Thus, there is only one possible value of k , 122.**

2) How many positive integers less than 200 have an odd number of positive integer divisors?

Solution:

Factors come in pairs in all cases except one, perfect squares. The question is thus asking us how many perfect squares there are between 0 and 200. A listing of the perfect squares is as follows: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196. There are **14** perfect squares less than 200.

3) The number $2^{48} - 1$ is exactly divisible by two numbers between 60 and 70. What are they?

Solution

Let's factor the expression:

$$\begin{aligned} 2^{48} - 1 &= (2^{24} - 1)(2^{24} + 1) \\ &= (2^{12} - 1)(2^{12} + 1)(2^{24} + 1) \\ &= (2^6 - 1)(2^6 + 1)(2^{12} + 1)(2^{24} + 1) \end{aligned}$$

Two of the factors listed above are **63 and 65**. Since it's given that there are only 2 divisors between 60 and 70, these must be the two.

4) Integers x and y with $x > y > 0$ satisfy $x + y + xy = 80$. What is x ?

Solution

Add 1 to both sides of the equation to yield:

$$xy + x + y + 1 = 81$$

$$(x+1)(y+1) = 81$$

If two integers multiply to 81 and they are distinct, those two integers must be either 1 and 81 or 3 and 27. But, if we set $y+1$ to 1, y would be 0 and this isn't allowed. It follows that $x+1 = 27$ and $y+1 = 3$. Thus, **$x = 26$** . (Also, we can deduce that $y = 2$.)