COT 3100 Recitation #5: Arithmetic, Geometric Series Quiz and Solutions
9/21-25/2020

1) Let the 10^{th} term of an arithmetic series be 125 and the 50^{th} term of the same arithmetic series be 45. What is the first term of the series?

Solution
First, let's find the common difference (let this be d) and let a_i be the i^{th} term of the sequence.

$a_{50} = a_{10} + (50-10)d$
$45 = 125 + 40d$
$-80 = 40d$
$d = -2$

The common difference is -2. It follows that

$a_{10} = a_1 + 9d$
$125 = a_1 + 9(-2)$
$a_1 = 125 + 18 =$ **143.**

2) Determine the sum of the infinite geometric series:

$15 + 6 + 2.4 + \ldots$

Solution
The first term of the series is 15 and the common ratio is $\frac{6}{15} = \frac{2}{5}$, which we obtain by dividing the second term by the first. The corresponding sum is $\frac{15}{1-\frac{2}{5}} = \frac{15}{\frac{3}{5}} = \mathbf{25}$.

3) Let $a_1, a_2, a_3, \ldots, a_{100}$ be the first 100 terms of an arithmetic series such that the sum of these 100 terms is 800. If the common difference of the sequence is 2, determine the following sum:

$$a_1 + a_3 + a_5 + a_7 + \ldots a_{99}$$

Solution
Since the common difference is 2, we can express $a_{2i} = a_{2i-1} + 2$, for each integer i from 0 to 49.

It follows that $a_2 + a_4 + a_5 + a_8 + \ldots a_{100} = (a_1 + 2) + (a_3 + 2) + \ldots (a_{99} + 2)$
$= (a_1 + a_3 + a_5 + a_7 + \ldots a_{99}) + 2(50)$

Thus, we have $a_1 + a_2 + a_3 + \ldots + a_{100} = 800$

Substitute for the even terms as shown above, and then solve for the desired sum:

$2(a_1 + a_3 + a_5 + a_7 + \ldots a_{99}) + 100 = 800$
$2(a_1 + a_3 + a_5 + a_7 + \ldots a_{99}) = 700$
$2(a_1 + a_3 + a_5 + a_7 + \ldots a_{99}) =$ **350**

4) The sequence 2^a, 4^b, 8^{a-b} is a geometric sequence. What is the ratio of a to b? (Note: a ratio is where we write the fraction equivalent to a/b in the form a:b where a and b are integers which don't share any common factors.)

Solution

Rewrite the sequence using a common base, 2:

$2^a, (2^2)^b, (2^3)^{a-b}$

$2^a, 2^{2b}, 2^{3(a-b)}$

For this to be a geometric sequence, the ratio of successive terms must be the same. The second term divided by the first is 2^{2b-a} and the third term divided by the second is $2^{3(a-b)-2b}$. (This is because when we divide two exponents with a common base, we just subtract exponents.) Since these are equal and the exponential function is monotonically increasing for a base greater than 1, we can equate the exponets:

$2b - a = 3(a-b) - 2b$
$2b - a = 3a - 3b - 2b$
$7b = 4a$
$\frac{a}{b} = \frac{7}{4}$

Thus, the ratio of a to b is **7:4.**

5) Ernie earns \$20 on his first day of work. For each subsequent day, he earns \$1 more than he earned the previous day. Maria starts working on the same day as Ernie and she earns \$5 for her first day of work. For each subsequent day, she earns \$3 more than she earned the previous day. After which day of work have both Ernie and Maria earned exactly the same amount of money in total?

Solution

Ernie's total pay after n days can be modeled as the sum of an arithmetic sequence with first term 20 and common difference 1. Maria's total pay after n days can be modeled as the sum of an arithmetic sequence with first term 5 and common difference 3. Let e_i represent Ernie's pay on day i and m_i represent Maria's pay on day i. Let $S_{e,n}$ represent the total of Ernie's pay for n days and let $S_{m,n}$ represent Maria's total pay for n days. Then we have the following equations for the sum of pay:

$S_{e,n} = \frac{(e_1+e_n)}{2} \times n, S_{m,n} = \frac{(m_1+m_n)}{2} \times n$

Set these equal to each other and solve for n:

$$\frac{(e_1 + e_n)}{2} \times n = \frac{(m_1 + m_n)}{2} \times n$$

Canceling, we have:

$$e_1 + e_n = m_1 + m_n$$

Now, use the formulas for calculating the nth term of an arithmetic sequence to determine e_n and m_n:

$$e_n = e_1 + (n - 1) = 20 + n - 1 = n + 19$$

$$m_n = m_1 + 3(n - 1) = 5 + 3(n - 1) = 5 + 3n - 3 = 3n + 2$$

Substituting, we have:

$$20 + n + 19 = 5 + 3n + 2$$

$$n + 39 = 3n + 7$$

$$2n = 32$$

$$n = \mathbf{16}$$