Roots of Polynomials (Week of 8/24 - 8/28)

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Standard polynomial something like $f(x) = 3x^3 + 7x^2 - 5$.

We have terms, each term is the variable (often times x), raised to some power. Each term also has a coefficient.

A root of a polynomial is a value that, when you plug it in for x, the value of the polynomial is 0.

Start with quadratics…

$x^2 - 5x + 6 = 0$

How do we find the roots?

1) Factoring
2) Quadratic Equation
3) Guess and Check using Synthetic Division (almost never used for quadratics, but used for higher degree polynomials)

Method #1

$x^2 - 5x + 6 = 0$
$(x - 3)(x - 2) = 0$, so now $x = 3$ or $x = 2$ are the roots.

Method #2

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$
$b = -5$
$c = 6$

$$= \frac{5 \pm \sqrt{25 - 4(1)(6)}}{2 \cdot 1}$$

$$= \frac{5 \pm 1}{2} = 3, 2$$

Method #3

1	-5	6
	1	-4
1	-4	2 (so 1 isn't root, but we know that f(1) = 2

1	-5	6
	2	-6
2	-3	0 (2 is a root)

When you do synthetic division, the value you get at the end is f(k), where k is the value you started with, to check if it's a root. If you get 0, you found a root.

The rational root theorem says that any rational root must be a divisor of the constant term divided by a divisor of the leading term.

Let's look at roots of a quadratic differently!!!

For now, assume a leading coefficient of 1:

$x^2 + bx + c = 0$

Let the roots of this equation be r and s. Then another way to write the same exact equation is:

$(x - r)(x - s) = 0$

So, if we want a relationship for b and c with r and s, set these equal:

A key idea in mathematics is find TWO different expressions for the same quantity and then set them equal to each other!!!

1) Polynomial view, with coefficients
2) Factor view, with roots

$x^2 + bx + c = (x - r)(x - s)$
$x^2 + bx + c = x^2 - rx - sx + rs$
$x^2 + bx + c = x^2 - (r + s)x + rs$

For these polynomials to be identical, each pair of corresponding coefficients must be the same, so $b = - (r+s)$, $c = rs$.

So, in a quadratic, the sum of the roots is the negative of the x coefficent and the product of the roots is that constant coefficient.

What is there is an a?

$$ax^2 + bx + c = 0$$
$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$
$$\text{so} \quad r + s = -\frac{b}{a}$$
$$rs = \frac{c}{a}.$$

Can we extend this to any degree polynomial?

Yes!

For example in the cubic equation $x^3 + ax^2 + bx + c = 0$, if the roots are r, s and t, we have:

a = -(r+s+t)
b = rs + rt + st
c = -rst

In this class, I'll just ask that you understand this relationship for quadratics and cubics.

Question #1

(2x + 3)(x - 4) + (2x + 3)(x - 6) = 0.
(2x + 3)(x - 4 + x - 6) = 0
(2x + 3)(2x - 10) = 0
x = -3/2 or x = 5
Sum of these is 7/2.

Question #2
Multiplying Polynomials

Question #2
Multiplying Polynomials

$(3x^3 + 2x - 4)(2x^4 - 5x^2 + x - 6)$

	$2x^4$	$-5x^2$	x	-6
$3x^3$	$6x^7$	$-15x^5$	$3x^4$	$-18x^3$
$2x$	$4x^5$	$-10x^3$	$2x^2$	$-12x$
-4	$-8x^4$	$20x^2$	$-4x$	24

Gather like terms: $6x^7 - 11x^5 - 5x^4 - 28x^3 + 22x^2 -16x + 24$

Question #3

Quadratic is $x^2 + 3x - 6$. Let its roots be r and s. Find the quadratic with leading coefficient 1 that has roots r/s and s/r.

We know that $r + s = -3$
$rs = -6$, using the sum and product of roots relationship

Now, consider our new equation with roots r/s and s/r. Let's find their sum and product:

sum = $r/s + s/r = (r^2 + s^2)/rs$

product = $(r/s)*(s/r) = 1$

So, we need to find out what $r^2 + s^2$ is and we must determine what rs is.

The latter is easy, we know it's -6 from our given equation.

What about finding $r^2 + s^2$?

Let's try this: $(r + s)^2 = r^2 + 2rs + s^2$, using the given info $r + s = -3$
$(-3)^2 = r^2 + s^2 + 2(-6)$
$9 = r^2 + s^2 - 12$
$21 = r^2 + s^2$

sum = r/s + s/r = ($r^2 + s^2$)/rs = 21/(-6) = -7/2

Thus, the desired quadratic is $x^2 + (7/2)x + 1 = 0$

Question #4

Find xyz.

$x + \frac{1}{y} = 4$

$y + \frac{1}{z} = 1$

$z + \frac{1}{x} = \frac{7}{3}$

$x + \frac{1}{y} + y + \frac{1}{z} + z + \frac{1}{x} =$
$4 + \frac{7}{3} + 1 = \frac{22}{3}$

$\left(x + \frac{1}{y}\right)\left(y + \frac{1}{z}\right)\left(z + \frac{1}{x}\right) = \frac{28}{3}$

$xyz + \underline{y + x} + \underline{\underline{\frac{1}{z}}} + \underline{z} + \underline{\underline{\frac{1}{y}}} + \underline{\underline{\frac{1}{x}}} + \frac{1}{xyz} = \frac{28}{3}$

$xyz + \frac{1}{xyz} + \frac{22}{3} = \frac{28}{3}$

$xyz + \frac{1}{xyz} = 2$

Let $T = xyz$.

$T + \frac{1}{T} = 2$

$$t^2 - 2t + 1 = 0$$

$$(t-1)^2 = 0$$

$$t = 1$$

Thus, xyz = 1.