# COT 3100 Fall 2020 Homework #10 Solutions

1) (14 pts) Let $R_1$ and $R_2$ be relations on a set A = {1, 2, 3, 4}.
In particular, let $R_1$ = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 4), (3, 4), (4, 2), (4, 3) (4, 4)} and
$R_2$ = {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (4, 1), (4, 2)}.

Determine the following:

a) Whether or not $R_1$ is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
b) Whether or not $R_2$ is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
c) The relation $R_1 \circ R_2$.
d) The relation $R_2 \circ R_1$.
e) $R_1 \cup R_2$
f) $R_1 \cap R_2$
g) The reflexive, symmetric and transitive closures of both $R_1$ and $R_2$.

**Solution**

a) $R_1$ is not reflexive since it does not contain (3, 3).
$R_1$ is not irreflexive since it contains (1, 1).
$R_1$ is symmetric, for each ordered pair of the form (a, b) it contains, it also contains (b, a).
$R_1$ is not anti-symmetric, since it contains both (1, 2) and (2, 1).
$R_1$ is not transitive because it contains (3, 4) and (4, 3), but doesn't contain (3, 3).

b) $R_2$ is not reflexive since it does not contain (1, 1).
$R_2$ is irreflexive since it doesn't contain any element of the form (a, a).
$R_2$ is not symmetric, because it contains (1, 3) but not (3, 1).
$R_2$ is not anti-symmetric, since it contains both (1, 2) and (2, 1).
$R_2$ is not transitive because it contains (1, 2) and (2, 1), but doesn't contain (1, 1).

c) $R_1 \circ R_2$ = {(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,4), (4,1), (4,2), (4,4)}

d) $R_2 \circ R_1$ = {(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (4,1), (4,2), (4,3)}

e) $R_1 \cup R_2$ = {(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,4), (3,4), (4,2), (4,3) (4,4)}

f) $R_1 \cap R_2$ = {(1,2), (2,1), (4,2)}

g) r($R_1$) = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 4), (3, 3), (3, 4), (4, 2), (4, 3) (4, 4)}
s($R_1$) = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 4), (3, 4), (4, 2), (4, 3) (4, 4)}
t($R_1$) = {(1, 1), (1, 2), (1,3), (1,4), (2, 1), (2, 2), (2,3), (2, 4), (3,1), (3,2), (3,3) (3, 4), (4,1), (4, 2), (4, 3) (4, 4)}

r($R_2$) = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 3), (4, 1), (4, 2), (4, 4)}.
s($R_2$) = {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1), (4, 2)}.
t($R_2$) = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (4, 1), (4, 2), (4, 3), (4, 4)}.

2) (5 pts) Let R be a relation over the positive integers defined as follows:

R = {(a,b) | 2b < a < 3b }

Determine whether or not R satisfies the following properties. Give a brief justification for each of your answers.

(i) reflexive
(ii) irreflexive
(iii) symmetric
(iv) anti-symmetric
(v) transitive

**Solution**
(i) No, for all a, 2a < a is false, therefore, for all a, (a, a) is not in R. Specifically, (1, 1) is not in R.

(ii) Yes, since for all positive integers a, 2a < a is false, for all a, (a, a) is not in R and R is irreflexive.

(iii) No. Note that (5, 2) is in R because 2(2) < 5 < 3(2), but that (2, 5) is not in R.

(iv) Yes. In no case is (a, b) and (b, a) in R. It is not possible for 2b < a and for 2a < b. This is a contradiction for all positive integers a and b.

(v) No. Note that (11, 5) is in R and (5, 2) is in R, but (11, 2) is NOT in R.

3) (6 pts) How many symmetric relations on the set A = {1, 2, 3, 4, 5, 6, 7, 8} contain the ordered pairs (2, 3), (3, 2), (4, 7), (5, 5) and (8, 7)?

**Solution**
If a relation is symmetric and contains (4, 7), it must also contain (7, 4).
Similarly, a symmetric relation that contains (8, 7) must also contain (7, 8).

This means the relations we want to count all contain the following elements:

(2, 3), (3, 2), (4, 7), (7, 4), (8, 7), (7, 8) and (5, 5)

There are 7 remaining ordered pairs of the form (a, a) and 64 - 8 - 6 = 50 ordered pairs of the form (a, b) where a $\neq$ b. These 50 ordered pairs can be grouped in 25 pairs where each pair contains both the ordered pair (a, b) and (b, a), for each a and b with a $\neq$ b, that isn't listed above.

For each of our 25 pairs of the form (a, b) and (b, a), either neither of the items in the ordered pair must be in the relation, or BOTH of the items in the ordered pair must be in the relation.

Thus, when creating a valid symmetric relation with (2, 3), (3, 2), (4, 7), (7, 4), (8, 7), (7, 8) and (5, 5), we have $2^{25}$ ways in which we can add in (or not add in) the pairs of terms previously designated. Each choice of add/not add for the 25 pairs are independent of one another so we can multiply the 2 choices for each of the pairs.

For the 7 ordered pairs of the form (a, a) not already in the relation, we can either add it or not add it to the relation, so there are $2^7$ ways to add ordered pairs of the form (a, a).

Each of these choices are independent of the other choices, thus, there are a total of $2^{25} \times 2^7 =$ **$2^{32}$** possible relations over the set A which contain the ordered pairs (2, 3), (3, 2), (4, 7), (5, 5) and (8, 7).

4) (8 pts) Let the relation R, over the set of 8-bit unsigned integers (0 to 255), be defined as

$$R = \{(a, b) | a \wedge b \ \equiv 0 \ (mod\ 8)\}$$

Note: The carat symbol represents a bitwise XOR. Prove that R is an equivalence relation. How many equivalence classes does R have? What would be an easier way of stating this exact same relation that doesn't use the bitwise XOR operator?

**Solution**

The bitwise OR of two items is equal to zero if and only if the two items themselves are equal. (This is because only for equal bits is the XOR equal to 0 and this would have to occur for each relevant bit.) In this case, if the bitwise XOR of a and b leaves a remainder of 0 when divided by 8, that means that the last three bits a and b must be the same, which is another way of saying that both are equivalent mod 8. Thus, we can re-express R as follows:

$$R = \{(a, b) | a \equiv b \ (mod\ 8)\}$$

R has 8 equivalence classes, one for possible integer mod 8, so each integer in between 0 and 7, inclusive.

5) (4 pts) Let $f(x) = x^2 + 4x - 45$ with a domain of all real $x \in [-\infty, -2]$. Prove that f is injective. What is the range of f? (You may either use Calculus or complete the square to prove your answers.)

**Solution**

To prove that f is injective, we must show that if f(a) = f(b), then a = b. Let's use direct proof here:

f(a) = f(b), with a, b ≤ -2

$a^2 + 4a - 45 = b^2 + 4b - 45$
$a^2 + 4a = b^2 + 4b$

$a^2 + 4a + 4 = b^2 + 4b + 4$
$(a + 2)^2 = (b + 2)^2$

The only way for two squares to be equal is if the second is either the positive of negative of the first, so we have:

$a + 2 = \pm(b + 2)$
$a = -2 \pm(b + 2)$

Which translates to either $a = -2 + b + 2$ OR $a = -2 - b - 2$
$a = b$ OR $a + b = -4$

Given that $a \leq -2$ and $b \leq -2$, the latter is ONLY possible if $a = -2$ and $b = -2$, but in this case, this infers that $a = b$ anyway. Thus, in all cases, we've proven that $a = b$, as desired and it follows that the function is injective.

6) (4 pts) Find $f^{-1}(x)$ for the function given in question #5.

**Solution**
$f(x) = x^2 + 4x - 45$

Swap x and y and solve for y:

$x = y^2 + 4y - 45$

$x + 49 = y^2 + 4y - 45 + 49$
$x + 49 = y^2 + 4y + 4$
$x + 49 = (y + 2)^2$

Now, take the square root of both sides, yielding:

$y + 2 = \pm\sqrt{x + 49}$
$y = -2 \pm \sqrt{x + 49}$

At this juncture, since we know the domain of the original function was $x \leq -2$, we know that the range of the inverse function must be the same. To achieve this, we must choose the negative sign for the plus/minus, yielding the following inverse function:

$f^{-1}(x) = -2 - \sqrt{x + 49}$

7) (4 pts) Let f(x) = $\sqrt{e^x}$ and g(x) = $x^2$. Determine $h_1(x)$ = f(g(x)) and $h_2(x)$ = g(f(x)). What are the largest possible domains for which functions $h_1(x)$ and $h_2(x)$ can be defined?

**Solution**

$h_1(x) = f(g(x)) = f(x^2) = \sqrt{e^{x^2}} = e^{(x^2)(\frac{1}{2})} = e^{\frac{x^2}{2}}$

$h_2(x) = g(f(x)) = g\left(\sqrt{e^x}\right) = (\sqrt{e^x})^2 = e^x$

The domains for both functions are all possible real numbers. One can square any real number, and e raised to any real number is non-negative, thus, the square root of that quantity can always be taken. An answer of complex numbers will also be accepted.

8) (5 pts) Please give a summary of the life and mathematical contributions of Emmy Noether. Please aim for a length of roughly 200 - 400 words. ***Your summary must be typed.*** Please state the sources you used in writing your summary.

**Sample Summary**

Amalie Emmy Noether was born in 1882 in Erlangen, Germany to a Jewish family. Her father, Max Noether, was a mathematician, leading Emmy to study mathematics at the University of Erlangen, where her father was a lecturer. After completing her dissertation, Noether went to work at the Mathematical Institute of Erlangen for seven years, where she made no income. In 1915, she joined the Universiry of Gottingen, with great resisstence from the faculty due to her being a woman. Mathematician B. L. van der Waerden become acquainted with Noether and use her as a source for a lot of the landmark material that he would go on to publish, such as his 1931 textbook, *Moderne Algebra*. Noether would go on to join the International Congress of Mathematicians in Zurich, becoming one of the most influential mathematicians in the world of algebra. Noether lost her position at the university due to the influence of Germany's Nazi government, after which she went to Bryn Mawr College in Pennsylvania.

During her time at Mathematical Institute of Erlangen, Nother made several contributions to algebraic invariants and number fields. She developed what is now known as Noether's theorem, which has become a staple theorem in theoretical physics and the calculus of variations. At the University of Gottingen, she made further contributions to theoretical physics when she published her paper, *Theory of Ideals in Ring Domains* in 1921, which led to mathematicians describing objects that satisfy an ascending or descending chain conditions as Noetherian. She published several papers on noncommutative algebra and hypercomplex numbers. Noether would also inspire several mathematicians and was credited in numerous papers by other mathematicians that covered a wide range of fields.

Source: https://scientificwomen.net/women/noether-emmy-75