COT 3100 Fall 2020 Homework #8 Solutions

1) (6 pts) Rolling two four-sided dice, what's the likelihood of getting the same sum twice in a row?

Solution
First, let's make a table of all possible rolls for two dice:

D1/D2	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

This gives us the following probabilities for each sum:

Sum	Probability
2	1/16
3	1/8
4	3/16
5	1/4
6	3/16
7	1/8
8	1/16

The probability that we roll any of these sums twice in a row is simply the square of the appropriate value on the right hand side of the table. Thus, to get the total probability that we roll the same sum twice, we just sum over the probabilities of getting each individual sum twice, which is the following:

$$2\left(\frac{1}{16}\right)^2 + 2\left(\frac{1}{8}\right)^2 + 2\left(\frac{3}{16}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{2+8+18+16}{16^2} = \frac{44}{256} = \mathbf{\frac{11}{64}}$$

2) (6 pts) Rolling three four-sided dice, what is the probability of totaling at least 9?

Solution

The probability of any ordered set of dice rolls is $(\frac{1}{4})^3 = \frac{1}{64}$. Here are the unordered set of dice rolls that total 9 or greater, as well as the number of different orderings for each set:

Unordered Set	Number of Orderings
4,4,4	1
4,4,3	3
4,4,2	3
4,4,1	3
4,3,3	3
4,3,2	6
3,3,3	1
Total	20

In general, the number of orderings is calculated using the permutation formula for objects when some objects are repeated. Since these sets are so small, one can either apply the formula or quickly count them by hand.

It follows that the desired probability is $\frac{20}{64} = \mathbf{\frac{5}{16}}$.

3) (6 pts) Rolling three four-sided dice twice, what is the probability of the second total being larger than the first?

Solution

First, we have to work out the probability distribution of getting each possible value for three dice:

We can build on our work for two dice, using that as one dimension of our table and the third die as the other dimension. Here though, we have to take care and multiply the probabilities of obtaining each sum to calculate the requisite probability. In the table, the red number represents the sum of the three dice and the blue number represents the probability of obtaining that particular sum in the specified order (the sum of the first two dice is the column label and the third die is the row label).

D3/D12	2	3	4	5	6	7	8
1	3, 1/64	4, 2/64	5, 3/64	6, 4/64	7, 3/64	8, 2/64	9, 1/64
2	4, 1/64	5, 2/64	6, 3/64	7, 4/64	8, 3/64	9, 2/64	10, 1/64
3	5, 1/64	6, 2/64	7, 3/64	8, 4/64	9, 3/64	10, 2/64	11, 1/64
4	6, 1/64	7, 2/64	8, 3/64	9, 4/64	10, 3/64	11, 2/64	12, 1/64

Now, we coalesce this data into a single chart showing the probabilities of obtaining each sum:

Sum	Probability
3	1/64
4	3/64
5	6/64
6	10/64
7	12/64
8	12/64
9	10/64
10	6/64
11	3/64
12	1/64

Unfortunately, there are many ways to obtain the second roll greater than the first roll. Let this probability be X. Notice that the probability of obtaining the first roll greater than the second roll is the same! The only third option is that both rolls are identical. Thus, we get the following equation when adding up disjoint probabilities:

P(first roll > second roll) + P(second roll > first roll) + P(first roll = second roll) = 1
X + X + P(first roll = second roll) = 1
2X = 1 - P(first roll = second roll)
$X = \frac{1-P(first\ roll=second\ roll)}{2}$

Thus, it suffices for us to find the probability that two consecutive rolls of 3 dice will result in the same total. This sum is:

$$2(\frac{1}{64})^2 + 2(\frac{3}{64})^2 + 2(\frac{6}{64})^2 + 2(\frac{10}{64})^2 + 2(\frac{12}{64})^2 = \frac{2(1 + 9 + 36 + 100 + 144)}{64^2} = \frac{145}{2^{10}}$$

It follows that our desired probability is $\frac{1-\frac{145}{1024}}{2} = \mathbf{\frac{879}{2048}}$.

4) (8 pts) The Tauroto region contains 31 different species of IceCream-type small monsters. Each capture of an IceCream-type small monster has an equal chance of being any of the 31 species.

a) If a collector captures five of these small monsters, what is the probability that they have captured at least two small monsters of the same species?

b) What is the lowest number of IceCream-type small monsters that a collector must capture to make capturing at least two of the same species more likely than not?

Solution

a) The probability all 5 are different is $\frac{31}{31} \times \frac{30}{31} \times \frac{29}{31} \times \frac{28}{31} \times \frac{27}{31} \sim .712$. It follows that the probability that at least two of the monsters are of the same species is roughly 1 - .712 = **.288.** The reasoning behind the product is that for each monster you catch, you multiply the probability that it's different than all previous monsters. Thus, when we calculate the third term, for example, we've already caught to monsters, so 29 out of the 31 types are different than the first two, so the probability the third monster is unique is 29/31.

b) We want the smallest possible value of k for which $\prod_{i=0}^{k-1} \frac{31-i}{31} < .5$, since we want the probability that all the monsters are of different species to be less than .5. By hand and calculator, there is no easier way than iteratively building the product. We can start with what is above and add a term to it:

$\frac{31}{31} \times \frac{30}{31} \times \frac{29}{31} \times \frac{28}{31} \times \frac{27}{31} \times \frac{26}{31} \sim .597$
$\frac{31}{31} \times \frac{30}{31} \times \frac{29}{31} \times \frac{28}{31} \times \frac{27}{31} \times \frac{26}{31} \times \frac{25}{31} \sim .482$

It follows that the minimum value k for which it's more than 50% likely for us to capture at least two monsters of the same species is **7.** If we capture 7 monsters, we have roughly a 51.8% chance of catching at least two of the same species.

5) (5 pts) The Knightrola Lottery has been updated to allow choosing 6 numbers out of 63 on a $1 ticket.

a) Choosing 3 numbers correctly gives a $10 prize.
b) Choosing 4 numbers correctly gives a $50 prize.
c) Choosing 5 numbers correctly gives a $5,000 prize.

If the Lottery expects to **precisely break even**, how much is its prize for getting all 6 numbers correct?

Solution

The probability of matching exactly k values is $\frac{\binom{6}{k}\binom{57}{6-k}}{\binom{63}{6}}$. The reasoning behind this was shown in class. In particular, the denominator is the # of ways to choose 6 values out of the 63 possible, representing the total number of possible tickets. The numerator represents the number of ways to select k of the correct number (out of 6 correct numbers) and 6-k of the incorrect number, out of the 57 incorrect numbers. We want to solve for X (winning prize) in the equation:

$$\frac{10\binom{6}{3}\binom{57}{3}}{\binom{63}{6}} + \frac{50\binom{6}{4}\binom{57}{2}}{\binom{63}{6}} + \frac{5000\binom{6}{5}\binom{57}{1}}{\binom{63}{6}} + \frac{X}{\binom{63}{6}} = 1$$

The left hand side is the expectation formula for the returns from the lottery, and since we are spending 1 dollar for a ticket and want the lottery to break even, this is the right hand side. The corresponding equation when we multiply through by $\binom{63}{6}$ is

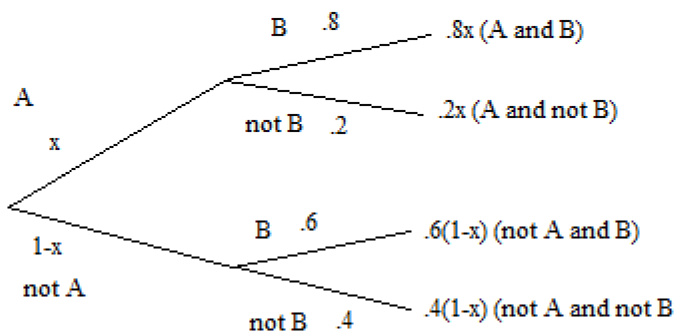
$$5852000 + 1197000 + 1710000 + X = 67945521$$

$$X = \$\mathbf{59,186,521}$$

Thus, to expect to break even, the prize for matching all of the numbers should be set at **$59,186,521.**

6) (6 pts) Let A and B be events such that $p(B\ |A\ ) = .8$, $p(\ \bar{B}|\bar{A}) = .4$, and $p(B) =\ .62$. What is $p(A)$?

Solution
Let's draw the corresponding probability tree, with the event A as the first branch, and letting x = the probability of event A:



From this tree, we can see that the probability of B is the sum of the probability of A and B and the probability of not A and B, which is .8x + .6(1-x). We also know this probability is 62%. Thus, we get the equation:

$$.8x + .6(1 - x) = .62$$

$$.8x + .6 - .6x = .62$$

$$.2x = .02$$

$$\boldsymbol{x = .1}$$

Thus, the probability of event A occurring is **10%.**

7) (8 pts) Arup's class roll has 30 students listed in alphabetical order. Exactly 5 of those students received As in the course. What is the probability that at least two people who were adjacent on the alphabetical list both received As?

Solution

The sample space is $\binom{30}{5}$, since that is the number of ways we can choose the 5 students out of 30 who received As in the course.

Of these possibilities, we must find which ones do NOT have any two As consecutive. This will allow us to find the complementary probability to the one that is being asked for, and then we can subtract from 1 to answer the question posed. (This is much easier than attempting to consider all the unique ways in which we can select at least two consecutive people the list.)

Now, we must find how many ways there are to select five slots out of 30 in a line with no two adjacent. There are 25 slots which represent the non A students. Use these 25 students as separators (we can denote these students as S_1 through S_{25}) and draw the gaps in between them:

__ S_1 __ S_2__ … __ S_{24} __ S_{25}__

There are 26 gaps in total and we can choose any 5 of those gaps to place the 5 A students, which we can do in $\binom{26}{5}$ ways.

It follows that the probability that no two As students are adjacent on the roll is $\frac{\binom{26}{5}}{\binom{30}{5}} = \frac{26\times25\times24\times23\times22}{30\times29\times28\times27\times26} = \frac{25\times24\times23\times22}{30\times29\times28\times27} = \frac{2530}{5481} \sim 46.2\%$

Thus, to answer the original question, the probability that some 2 students who both received As are next to each other on the roll is $\mathbf{\frac{2951}{5481}} \sim 53.8\%$

8) (5 pts) Give a summary of mathematician Melanie Wood's life and professional accomplishments, thus far. Please aim for a length of roughly 200 - 400 words. ***Your summary must be typed.*** Please state the sources you used in writing your summary.

Sample Write Up

Melanie Wood grew up in Indiana and earned recognition by being the first female to represent the United States of America in the 1998 and 1999 International Mathematics Olympiad. In both events she earned Silver Medals. She attended Duke University and continued to study mathematics and became the first American woman to be named a Putnam Fellow. After spending two years at Cambridge University, Wood attended Princeton University and earned her Ph. D. in 2009, studying under Manjul Bargava. Since Princeton, Wood has served as a mathematics professor at Stanford University (2009-2011), University of Wisconsin-Madison (2011-2018) and UC-Berkeley (2018-Present), though currently she is on leave from Berkeley and is at Harvard University. Dr. Wood's work is supported by both an NSF Career Grant and a Packard Fellowship for Science and Engineering.

Wood primarily studies number theory, with an interest in studying how many number fields there are as well as what sort of properties randomly chosen number fields might have. When it relates to number theory, Wood also studies probability, since she looks for ways to generate random fields and wants to know how often those fields have certain properties. She is the editor of the following journals: Algebra and Number Theory, International Mathematics Research Notices and Research in Number Theory. In addition to her research work, Dr. Wood has volunteered her time to help coach the United States IMO team (2005) and more recently participated in programs such as STEM Gems, to encourage young women to pursue STEM fields.

Sources

https://math.berkeley.edu/~mmwood/

https://en.wikipedia.org/wiki/Melanie_Wood

https://www.ams.org/journals/notices/201605/rnoti-p524.pdf

https://stemgemsbook.com/meet-our-gems/melanie-matchett-wood/