COT 3100 Fall 2020 Homework #7 Solutions

1) (9 pts) After catching sight of a particularly adorable online video, the Small Monster collector has begun a second career as a Small Monster researcher, intending to develop a taxonomic argument that marmosets are not in fact conventional animals, but are in fact Small Monsters. As a warm-up to their argument, they are reminding themselves of the biological family to which marmosets belong.

a. Taking into account identical letters, how many ways can the researcher arrange the word CALLITRICHIDAE?
b. Taking into account identical letters, how many ways can the researcher arrange the word CALLITRICHIDAE that start or end with vowels?
c. Taking into account identical letters, how many ways can the researcher arrange the word CALLITRICHIDAE that contain the substring LITERAL?

Solution

a) 14 letters; three I's; two A's, C's and L's; one D, E, H, R and T. $\frac{14!}{3!2!2!2!}$.

b) Since identical letters don't matter, we have three cases of starting with a vowel.

- A<CLLITRICHIDAE>: 13 letters; three I's; two C's and L's; one A, D, E, H, R and T. $\frac{13!}{3!2!2!} = 259{,}459{,}200$.
- E<CALLITRICHIDA>: 13 letters; three I's; two A's, C's and L's; one D, H, R and T. $\frac{13!}{3!2!2!2!} = 129{,}729{,}600$.
- I<CALLTRICHIDAE>: 13 letters; two A's, C's, I's and L's; one D, E, H, R and T. $\frac{13!}{2!2!2!2!} = 389{,}188{,}800$.

Ending with a vowel comes up with the same numbers. So, we sum up to:

$$(259{,}459{,}200 + 129{,}729{,}600 + 389{,}188{,}800) \times 2 = 1{,}556{,}755{,}200$$

…but we're only *almost* done. The inclusion-exclusion principle applies, and our current result double-counts results that both start and end with vowels. A and I repeat, but E doesn't, so we have the following cases:

- A<CLLITRICHIDE>A: 12 letters; three I's; two C's and L's; one D, E, H, R and T. $\frac{12!}{3!2!2!} = 19{,}958{,}400$.
- A<CLLITRICHIDA>E: 12 letters; three I's; two C's and L's; one A, D, H, R and T. $\frac{12!}{3!2!2!} = 19{,}958{,}400$.
- A<CLLITRCHIDAE>I: 12 letters; two I's, C's and L's; one A, D, E, H, R and T. $\frac{12!}{2!2!2!} = 59{,}875{,}200$.
- E…A: Same as A…E. $19{,}958{,}400$.
- E<CALLTRICHIDA>I: 12 letters; two A's, C's, I's and L's; one D, H, R and T. $\frac{12!}{2!2!2!2!} = 29{,}937{,}600$.

- I…A: Same as A…I. 59,875,200.
- I…E: Same as E…I. 29,937,600.
- I<CALLTRCHIDAE>I: 12 letters; two A's, C's, and L's; one D, E, H, I, R and T. $\frac{12!}{2!2!2!} = 59{,}875{,}200$.

The sum of those is $19{,}958{,}400 \times 3 + 29{,}937{,}600 \times 2 + 59{,}875{,}200 \times 3 = 299{,}376{,}000$. So our final answer is $1{,}556{,}755{,}200 - 299{,}376{,}000 = \mathbf{1{,}257{,}379{,}200}$.

Note: An alternative solution takes all permutations and subtracts out the ones that start and end with a consonant. Without fully going into the breakdown, for grading purposes, that approach leads to a result of the following form:

$$\frac{14!}{3!\,2!\,2!\,2!} - 2\left(\frac{12!}{3!\,2!\,2!}\right) - 2\left(\frac{12!}{3!\,2!}\right) - 16\left(\frac{12!}{3!\,2!\,2!}\right) - 12\left(\frac{12!}{3!\,2!\,2!\,2!}\right)$$

If one didn't have a calculator this breaks down to $\frac{\mathbf{14!}}{\mathbf{48}} - \frac{\mathbf{7}}{\mathbf{6}}(\mathbf{12!})$.

c) Make LITERAL a super-letter. Then we have CICHIDA as the remaining word. So 8 letters; two C's and I's; one A, D and H and LITERAL. $\frac{\mathbf{8!}}{\mathbf{2!2!}}$.

2) (2 pts) While they were researching marmosets, a particularly convoluted walk through Wikipedia caused the researcher to develop a fascination with the politics of religion in England in the 18th century. Taking into account identical letters, how many ways are there to arrange the letters in the word ANTIDISESTABLISHMENTARIANISM?

Solution

28 letters; five I's; four A's and S's; three N's and T's; two E's and M's; and one B, D, H, L and R.

$$\frac{\mathbf{28!}}{\mathbf{5!\,4!\,4!\,3!\,3!\,2!\,2!}}.$$

3) (15 pts) A Small Monster collector is visiting the Knightrola region, bringing 30 Small Monster Containment Devices with them. The devices are highly reliable and can effectively be assumed to always work. **How many different types of Small Monsters can the collector capture under the following conditions?**

a. The collector has access to Grass, Space, Electric Scooter, Mask, Citrus, and Rubber Duck type Small Monsters
b. The above, and intends to capture *at least* one of each of those types
c. Both of the above, and intends to capture *at least* five Space types
d. All of the above, and intends to capture *at most* four Electric Scooter types
e. All of the above, and intends to capture *at most* three Rubber Duck types

Solution

a) This is a combinations with repetition problem with n = 30 items to distribute amongst r = 6 types, thus, the number of ways to do this is:

$$\binom{n+r-1}{r-1} = \binom{6+30-1}{6-1} = \binom{35}{5}$$

b) In this version, there is no freedom to pick the first six items (one of each type), thus, we have n = 30-6 = 24 items to freely distribute amongst 6 types, which we can do in this many ways:

$$\binom{n+r-1}{r-1} = \binom{6+24-1}{6-1} = \binom{29}{5}$$

c) Compared to the previous part, we must capture 4 more Space types, so our value for n for which we can freely distribute amongst the 6 types is n = 24 - 4 = 20. We can do this in this many ways:

$$\binom{n+r-1}{r-1} = \binom{6+20-1}{6-1} = \binom{25}{5}$$

d) **Subtract** the combinations where we capture more than three (at least four) additional Electric Scooter types with our 20 SMCDs. (Remember we're already capturing one!) That's like capturing any type of monster with 16 SMCDs.

$$\binom{n+r-1}{r-1} = \binom{6+16-1}{6-1} = \binom{21}{5}$$

It follows that our final result is $\binom{25}{5} - \binom{21}{5}$.

e) **Subtract** the combinations where we capture more than two (at least three) additional Rubber Duck types with our 20 SMCDs. That's like capturing any type of monster with 17 SMCDs.

$$\binom{n+r-1}{r-1} = \binom{6+16-1}{5} = \binom{22}{5}$$

But wait – we just double-subtracted the cases where we capture more than three extra Electric Scooter types and more than two extra Rubber Duck types! That corresponds to capturing any type of monster with 13 SMCDs—

$$\binom{n+r-1}{r-1} = \binom{6+13-1}{6-1} = \binom{18}{5}$$

…and now we have to "add" those back. (Don't make the mistake of feeling we're really "adding" the double-count – we're ***subtracting it from the subtraction*** because, per the inclusion-exclusion principle, it isn't a thing.)

$$\binom{25}{5} - \binom{21}{5} - \binom{22}{5} + \binom{18}{5}$$

4) (4 pts) A tournament has n teams, ranked 1 to n. In the first match, the n^{th} ranked team plays the $(n-1)^{th}$ ranked team. The loser of that match finishes in last place. The winner of that match plays the $(n-2)^{th}$ ranked team. The loser of that match finishes in second to last place. The winner of that match goes on to play the $(n-3)^{th}$ ranked team, and so forth. The last match will pit the previous match winner against the first ranked team. The winner of this last match wins the tournament and the loser of this last match places second in the tournament. In how many different orders can the n teams finish. (Note: count two orders as different if at least one team receives a different finishing position in the two orders.) As a quick example, in a tournament with three teams, A, B and C, with A ranked first, B ranked second and C ranked third, here are the following four possible finishing orders: (A, B, C), (A, C, B), (B, A, C) and (C, A, B). Note that A can't finish worse than second place, since A only plays in the last match and is guaranteed to be first or second.

Solution

There are n-1 matches to be played and each match has one of two possible outcomes. Thus, the number of unique outcomes the matches can have all together is 2^{n-1}, the total number of possible strings of n-1 characters that are all either L (for loss) or W (for win), where loss/win represents the outcome for the challenger. (The challenger is the lower ranked player in the match.) Each of the different possible sets of match outcomes leads to a different final ranking and vice versa, thus the final answer to the question is just $\mathbf{2^{n-1}}$**.**

5) (4 pts) How many integers in the range 1 to 100,000 inclusive are divisible by 6, 15 or 21?

Solution

We use the Inclusion-Exclusion Principle, adding in the # of values divisible by each of the listed items, then subtracting out the # of items divisible by 6 and 15, the number of items divisible by 6 and 21 and the number of items divisible by 15 and 21. We finally add back in the number of items divisible by all 3. In general, the number of items divisible by a set of integers $a_1, a_2, \ldots, a_k$ from the first n positive integers is just $\left\lfloor \frac{n}{lcm(a_1,a_2,\ldots,a_k)} \right\rfloor$. Using this result, we get:

$$\left\lfloor \frac{100000}{6} \right\rfloor + \left\lfloor \frac{100000}{15} \right\rfloor + \left\lfloor \frac{100000}{21} \right\rfloor - \left\lfloor \frac{100000}{30} \right\rfloor - \left\lfloor \frac{100000}{42} \right\rfloor - \left\lfloor \frac{100000}{105} \right\rfloor + \left\lfloor \frac{100000}{210} \right\rfloor$$

If we simplify this, we get **21,904.**

6) (4 pts) A school's Student Government Association (SGA) has four specific positions: President, Vice President, Treasurer and Secretary. In addition, the, SGA has 12 members chosen at large. All of the seats must be filled by students from the SGA class, of which there are 50. No student may hold more than 1 position. How many different ways are there to fill the positions? (Note: Two ways are considered different if there is a person in one way who has a ***different role*** or no role in the other way. For example, if Jamila is President in one way and she is Vice President in another, those two ways must count differently. But if she is an at large member in both ways, that is not proof that those ways should be counted differently.)

Solution

There are 50 choices for President.
There are 49 choices for Vice President.
There are 48 choices for Treasurer.
There are 47 choices for Secretary.
From here, we can choose any 12 out of the remaining 46 students in the course to fill the members at large. We can do this in $\binom{46}{12}$ ways.

Since each of these choices is independent of the others, the total number of ways we can fill the positions is $\mathbf{\frac{50!}{46!}\binom{46}{12}}$**.** This can also be written as ${}_{50}P_4$ x ${}_{46}C_{12}$.

7) (7 pts) While the Small Monster collector, researcher, and ecclesiastical historian was busy with marmosets, we showed in class that we can an assign an integer order to the members of an infinite two-dimensional array; and thus, that infinite two-dimensional arrays are countable, and can be re-ordered as infinite one-dimensional arrays. Using that result, prove by induction that for any positive integer n, an infinite n-dimensional array is countable.

<u>Solution</u>
Base case: n = 1, an ordered set (infinite one dimensional array) is countable, based on the definition of countability.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k, that an infinite k dimensional array is countable.

Inductive step: Prove, for n = k+1, that a k+1 dimensional array is countable.

To prove that a set is countable, we must show a one to one correspondence between the natural numbers and the set.

The set we must prove is countable is a k+1 dimensional array. We can prove this set is countable by showing that the indexes into the set are countable. One way to consider this array is that it is the set of values of the form $(a_1, a_2, a_3, \ldots, a_{k+1})$, where a_1 is the first index into the array, a_2 is the second index into the array, and so forth.

For a moment, reconsider listing these indexes as follows: $((a_1, a_2, \ldots, a_k), a_{k+1})$. Namely, a set of ordered pairs, where the first item is a k-tuple of positive integers and the second item is a positive integer. There is a one to one correspondence between the first set of items and these items. (Just group together the first k items in any element of the first set to create the first item in the ordered pair in the second set.) Thus, if we can show that items of this latter form are countable, we will succeed in proving the items of the former form are countable.
From the inductive hypothesis, we know that all items of the form $(a_1, a_2, \ldots, a_k)$ are countable. Let's relabel these as the infinite one dimensional array $x_1, x_2, x_3, \ldots$, where x_1 is the first k-tuple of positive integers, x_2 is the second k-tuple of positive integers, and so forth. Now, we must show that the Cartesian Product of this set and the set of natural numbers (valid indexes into the k+1st dimension, are countable. We prove this by listing the items in a two dimensional grid, and ordering them via diagonal, going from the bottom left to the top right. We traverse the diagonals in order of the sum of their row and column.

Here is a portion of the diagram to show that the one to one correspondence exists:

X/N	1	2	3	4	5	6	7
x_1	$(x_1,1)^{1}$	$(x_1,2)^{3}$	$(x_1,3)^{6}$	$(x_1,4)^{10}$	$(x_1,5)^{15}$	$(x_1,6)^{21}$	$(x_1,7)^{28}$
x_2	$(x_2,1)^{2}$	$(x_2,2)^{5}$	$(x_2,3)^{9}$	$(x_2,4)^{14}$	$(x_2,5)^{20}$	$(x_2,6)^{27}$	$(x_2,7)$
x_3	$(x_3,1)^{2}$	$(x_3,2)^{8}$	$(x_3,3)^{13}$	$(x_3,4)^{19}$	$(x_3,5)^{26}$	$(x_3,6)$	$(x_3,7)$
x_4	$(x_4,1)^{7}$	$(x_4,2)^{12}$	$(x_4,3)^{18}$	$(x_4,4)^{25}$	$(x_4,5)$	$(x_4,6)$	$(x_4,7)$
x_5	$(x_5,1)^{11}$	$(x_5,2)^{17}$	$(x_5,3)^{24}$	$(x_5,4)$	$(x_5,5)$	$(x_5,6)$	$(x_5,7)$
x_6	$(x_6,1)^{16}$	$(x_6,2)^{23}$	$(x_6,3)$	$(x_6,4)$	$(x_6,5)$	$(x_6,6)$	$(x_6,7)$
x_7	$(x_7,1)^{22}$	$(x_7,2)$	$(x_7,3)$	$(x_7,4)$	$(x_7,5)$	$(x_7,6)$	$(x_7,7)$

The numbers in red represent the ordering of the Cartesian Product between the set x, which represents a k-dimensional array indexes and the natural numbers, which represent the $(k+1)^{st}$ dimension. Since this Cartesian Product is countable, and it is in a one to one mapping with the valid indexes to a (k+1)-dimensional array, it follows that a (k+1) dimensional array is countable as well.

For example, if $k = 2$, we would have $x_1 = (1, 1)$, $x_2 = (2, 1)$, $x_3 = (1, 2)$, $x_4 = (3, 1)$, $x_5 = (2, 2)$, $x_6 = (1, 3)$ and $x_7 = (4, 1)$. This would then mean that the ordering for a 3 dimensional array would start as follows:

(1, 1, 1), (2, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 2), (1, 1, 3), (3, 1, 1), (1, 2, 2), (2, 1, 3), (1, 1, 4), (2, 2, 1), (3, 1, 2), (1, 2, 3), (2, 1, 4), (1, 1, 5), (1, 3, 1), (2, 2, 2), (3, 1, 3), (1, 2, 4), (2, 1, 5), (1, 1, 6), (4, 1, 1), (1, 3, 2), (2, 2, 3), (3, 1, 4), (1, 2, 5), (2, 1, 6), (1, 1, 7), …

8) (5 pts) Give a summary of the academic contributions of Georg Cantor. Please aim for a length of roughly 200 - 400 words. ***<u>Your summary must be typed.</u>*** Please state the sources you used in writing your summary.

Sample Summary

Georg Cantor was born in Saint Petersburg, Russia in 1845. His father was a member of the Saint Petersburg stock exchange, and was able to pass a large inheritance onto Georg, who was able to pursue his passion of mathematics and raise a large family without worrying about money. Georg's family moved to Germany when he was 11 and he did most of his schooling in this area of Europe, attending Swiss Federal Polytechnique, the University of Berlin and the University of Gottingen.

In 1869, Cantor got a faculty position at the Halle University, where he stayed for the rest of his life. Part of the reason he didn't move to another university is that his ground breaking ideas in set theory created feuds with several rivals, such as Leopold Kronecker, one of his former teachers, and those rivals prevented him from getting positions at other universities.

Cantor advanced counting theory and set theory a great deal. He pioneered the use of the one-to-one correspondence to help answer both finite and infinite counting questions. He utilized the idea to show that the Cartesian Product of two countable sets was also countable, but that the number of real numbers and the number of subsets of the natural numbers were NOT countable, via an argument now known as Cantor Diagonalization. These ideas challenged some strongly held beliefs of contemporary mathematicians and philosophers of Cantor's day. Most of Cantor's best known work was conducted between 1874 and 1884. After that, he dealt with several bouts of depression and some family tragedies. Even though he had rivals, by the turn of the century, he had received a number of accolades and a number of mathematicians respected his work greatly. Cantor founded the German Mathematical Society, convened the first International Congress of Mathematicians, he received an honorary degree from University of St. Andrews, and David Hilbert was a strong proponent of his work.

Source

https://en.wikipedia.org/wiki/Georg_Cantor