COT 3100 Fall 2020 Homework #6
Please Consult WebCourses for the due date/time

1) Prove by induction on n that, for all positive integers n:

$$\sum_{i=1}^{n} \big(i(i!)\big) = (n+1)! - 1$$

Solution

Base case: n = 1, LHS = $\sum_{i=1}^{1} \big(i(i!)\big) = 1 \times 1! = 1$, RHS = $(1+1)! - 1 = 2! - 1 = 2 - 1 = 1$
Thus, the given assertion is true for n=1 and the base case holds.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{k} \big(i(i!)\big) = (k+1)! - 1$$

Inductive step: Prove for n = k+1 that

$$\sum_{i=1}^{k+1} \big(i(i!)\big) = ((k+1)+1)! - 1 = (k+2)! - 1$$

$$\sum_{i=1}^{k+1} \big(i(i!)\big) = \left[\sum_{i=1}^{k} \big(i(i!)\big)\right] + (k+1)((k+1)!)$$

$= (k+1)! - 1 + (k+1)((k+1)!)$, using the inductive hypothesis.

$$= (k+1)!\,[1+k+1] - 1$$

$$= (k+1)!\,[k+2] - 1$$

$$= (k+2)! - 1$$

This proves the inductive step. We can conclude for all positive integers n that

$$\sum_{i=1}^{n} \big(i(i!)\big) = (n+1)! - 1$$

Note: Since a factorial represents the product of consecutive integers starting at 1, when we multiply (k+1)! with k+2, the resulting product is just the product of all consecutive integers from 1 to k+2, which is just (k+2)!

2) Prove by induction on n that, for all positive integers n:

$$\sum_{i=1}^{n} i6^i = \frac{6(5n6^n - 6^n + 1)}{25}$$

<u>Solution</u>

Base case: n=1, LHS = $\sum_{i=1}^{n} i6^i = 1(6^1) = 6$, RHS = $\frac{6(5(1)6^1 - 6^1 + 1)}{25} = \frac{6(30-6+1)}{25} = 6$

Thus, the given assertion is true for n=1 and the base case holds.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{k} i6^i = \frac{6(5k6^k - 6^k + 1)}{25}$$

Inductive step: Prove for n = k+1 that

$$\sum_{i=1}^{k+1} i6^i = \frac{6(5(k+1)6^{k+1} - 6^{k+1} + 1)}{25}$$

$$\sum_{i=1}^{k+1} i6^i = \left(\sum_{i=1}^{k} i6^i\right) + (k+1)6^{k+1}$$

$= \frac{6(5k6^k - 6^k + 1)}{25} + \frac{25(k+1)6^{k+1}}{25}$, using the IH.

$$= \frac{5k6^{k+1} - 6^{k+1} + 6}{25} + \frac{25k6^{k+1} + (25)6^{k+1}}{25}$$

$$= \frac{30k6^{k+1} + 24(6^{k+1}) + 6}{25}$$

$$= \frac{30k6^{k+1} + 30(6^{k+1}) - 6(6^{k+1}) + 6}{25}$$

$$= \frac{6(5k6^{k+1} + 5(6^{k+1}) - (6^{k+1}) + 1)}{25}$$

$$= \frac{6(5k6^{k+1}(k+1) - (6^{k+1}) + 1)}{25}$$

This completes the proof of the inductive step. Therefore we can conclude for all positive integers n that $\sum_{i=1}^{n} i6^i = \frac{6(5n6^n - 6^n + 1)}{25}$, as desired.

3) Prove by induction on *n* that, for all positive integers *n*, $21|(4^{n+1}+5^{2n-1})$.

Solution
Base case: n = 1, $4^{1+1}+5^{2(1)-1}=16+5=21$. Since 21 = 1 x 21, it follows that 21 | 21 and that the base case holds.

Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$$21|(4^{k+1}+5^{2k-1}).$$

Namely, assume that there exists an integer c such that $4^{k+1}+5^{2k-1}=9c$.

Inductive step: Prove for n = k+1 that $21|\left(4^{(k+1)+1}+5^{2(k+1)-1}\right)$. In other words, show that there exists some integer d such that

$$4^{(k+1)+1}+5^{2(k+1)-1}=4^{k+2}+5^{2k+1}=21d$$

$$\begin{aligned}4^{(k+1)+1}+5^{2(k+1)-1} &= 4^{k+2}+5^{2k+1}\\ &= 4(4^{k+1})+(5^2)(5^{2k-1})\\ &= 4(4^{k+1})+(4+21)(5^{2k-1})\\ &= 4(4^{k+1})+(4)(5^{2k-1})+21(5^{2k-1})\\ &= 4(4^{k+1}+\ 5^{2k-1})+21(5^{2k-1})\\ &= 4(21c)+21(5^{2k-1}), \text{ using the IH.}\\ &= 21(4c+5^{2k-1})\end{aligned}$$

Since c is an integer and k is a positive integer, it follows that $4c+5^{2k-1}$ is an integer. Thus, we have shown that$4^{(k+1)+1}+5^{2(k+1)-1}$ is divisible by 21, completing the inductive step. It follows that for all positive integers n, $21|(4^{n+1}+5^{2n-1})$.

4) Let t_n be a sequence defined as follows: $t_0 = 5$, $t_1 = 12$, and for all $n \geq 2$, $t_n = 5t_{n-1} - 6t_{n-2}$. Using strong induction on n with 2 base cases, prove that for all non-negative integers n,

$$t_n = 3(2^n) + 2(3^n).$$

Solution
Base cases:

n=0, LHS = $t_0 = 5$, RHS = $3(2^0) + 2(3^0) = 3 + 2 = 5$. Thus the formula is true for n = 0.

n=1, LHS = $t_1 = 12$, RHS = $3(2^1) + 2(3^1) = 6 + 6 = 12$. Formula is true for n = 1 also.

Thus, both case cases are true and the formula is valid for n = 0, 1.

Inductive hypothesis: Assume for all non-negative integers n ≤ k, where k is an arbitrarily chosen positive integer, that $t_n = 3(2^n) + 2(3^n)$.

Inductive step: Prove for n = k + 1 that $t_{k+1} = 3(2^{k+1}) + 2(3^{k+1})$.

$t_{k+1} = 5t_k - 6t_{k-1}$

$= 5[3(2^k) + 2(3^k)] - 6[3(2^{k-1}) + 2(3^{k-1})]$, using the inductive hypothesis twice.

$= 15(2^k) + 10(3^k) - 18(2^{k-1}) - 12(3^{k-1})$

$= 15(2^k) + 10(3^k) - 9(2)(2^{k-1}) - 4(3)(3^{k-1})$

$= 15(2^k) + 10(3^k) - 9(2^k) - 4(3^k)$

$= 6(2^k) + 6(3^k)$

$= 3(2)(2^k) + 2(3)(3^k)$

$= 3(2^{k+1}) + 2(3^{k+1})$

This completes the inductive step. We can conclude that for all non-negative integers n, $t_n = 3(2^n) + 2(3^n)$.

Note: When we plugged into the inductive hypothesis, it's important to note that the smallest case for which we were plugging in was k-1. Since k is positive k-1 ≥ 0, so the inductive hypothesis applies.

5) Using induction on *n*, prove for all positive integers n that $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{bmatrix}$, where F_n denotes the n^{th} Fibonacci number. (Recall that the Fibonacci numbers are defined as follows: $F_0 = 0$, $F_1 = 1$, and for all n > 1, $F_n = F_{n-1} + F_{n-2}$.)

Solution

Base case: n = 1, LHS = $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, RHS = $\begin{bmatrix} F_{2(1)+1} & F_{2(1)} \\ F_{2(1)} & F_{2(1)-1} \end{bmatrix} = \begin{bmatrix} F_3 & F_2 \\ F_2 & F_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Thus the assertion is true for n = 1 and the base case holds.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^k = \begin{bmatrix} F_{2k+1} & F_{2k} \\ F_{2k} & F_{2k-1} \end{bmatrix}.$$

Inductive step: Prove for n = k+1 that

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} F_{2(k+1)+1} & F_{2(k+1)} \\ F_{2(k+1)} & F_{2(k+1)-1} \end{bmatrix} = \begin{bmatrix} F_{2k+3} & F_{2k+2} \\ F_{2k+2} & F_{2k+1} \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^k \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} F_{2k+1} & F_{2k} \\ F_{2k} & F_{2k-1} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2F_{2k+1} + F_{2k} & F_{2k+1} + F_{2k} \\ 2F_{2k} + F_{2k-1} & F_{2k} + F_{2k-1} \end{bmatrix}$$

$$= \begin{bmatrix} F_{2k+1} + F_{2k+2} & F_{2k+2} \\ F_{2k} + F_{2k+1} & F_{2k+1} \end{bmatrix}$$

$$= \begin{bmatrix} F_{2k+3} & F_{2k+2} \\ F_{2k+2} & F_{2k+1} \end{bmatrix}$$

This completes the inductive step. We can conclude that for all positive integers n, that $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{bmatrix}$.

Note: In each of the algebra steps, we applied the Fibonacci recurrence, coupling two consecutive Fibonacci numbers and adding them together to get the next one.

6) Give a summary of the academic contributions of Grigori Perelman. Please aim for a length of roughly 200 - 400 words. ***Your summary must be typed.*** Please state the sources you used in writing your summary.

Sample Summary

Grigori Perelman is a Russian mathematician who is best known for proving the Poincare Conjecture. Though many mathematicians shy away from the limelight, Perelman, even amongst mathematicians, is reclusive and prefers not to claim prizes for his work. Though he was awarded two of the most prestigious prizes in Mathematics: The Clay Mathematics Institute Millennium Prize and the Fields Medal.

Perelman was born in Leningrad Russia in 1966. His mother stopped graduate work in mathematics when he was born and recognized his unique talent for mathematics, making sure he was enrolled in the top mathematics programs in Leningrad while growing up. In 1982, Perelman represented the Soviet Union in the International Mathematics Olympiad (IMO), earning a Gold Medal and perfect score. The IMO is a world-wide competition for high school students where each country brings a team of their six best mathematics students. Perelman continued studying mathematics, earning his Ph.D. in 1990, and took a faculty position at the Leningrad Department of Steklov Institute of Mathematics. His top notch work in topology gave him opportunities to do research in the United States. He did research fellowships at both Stony Brook University in New York and UC-Berkeley. In 1994, while in the United States, Perelman proved the soul conjecture for Riemannian geometry. Soon thereafter, even though he received several faculty offers from prestigious American universities, Perelman decided to go back to Russia and work at the Steklov Institute of Mathematics.

After returning to Russia, Perelman started working on a technique called Ricci flow, developed by mathematician Richard Hamilton, in the hopes of proving the Poincare Conjecture. The conjecture states that, "Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere." Hamilton longed to prove the conjecture, but was stuck with a few sticky situations in his proof dealing with singularities. The term "homeomorphic" roughly means, "the same in general structure." For example, if I take a donut shaped stuffed animal and twist it, it might look like a pretzel stick. Even if these two shapes look different, one can be made to look like the other by untwisting it, so the two shapes are homeomorphic. Ultimately, Perelman used a technique called Ricci flow with surgery to deal with the outlying cases that Hamilton had difficulty with, to complete the proof. Perelman published his proof in three papers in 2002 and 2003. It took the mathematical community about three years to verify his proof and in 2006, the proof was accepted and Perelman was awarded the Fields medal. He refused the medal because he didn't believe he deserved it. He felt that Hamilton had done equal or more work to him in terms of proving the Poincare Conjecture. To him, he just cared that the proof was valid and accepted by others in the community, but he didn't want any of the hoopla surrounding it. In fact, it is said that Perelman quit mathematics in 2006 due to this notoriety that he explicitly wanted to avoid. In an article in the New Yorker, Perelman is quoted as saying, "As long as I was not conspicuous, I had a choice. Either to make some ugly thing or, if I didn't do this kind of thing, to be treated as a pet. Now, when I become a very conspicuous person, I cannot stay a pet and say nothing. That is why I had to quit."

To date, the Poincare Conjecture is the only problem of the seven Millennium Prize Problems to have been solved.

Sources
https://en.wikipedia.org/wiki/Grigori_Perelman

https://en.wikipedia.org/wiki/Poincar%C3%A9_conjecture