Fall 2020 COT 3100 Homework #2 Solutions

1) (4 pts) Prove the following set equality properties by membership table:

$$(A - C) \cap (C - B) = \emptyset.$$

A	B	C	$A - C$	$C - B$	$(A - C) \cap (C - B)$	$\emptyset$
0	0	0	0	0	0	0
0	0	1	0	1	0	0
0	1	0	0	0	0	0
0	1	1	0	0	0	0
1	0	0	1	0	0	0
1	0	1	0	1	0	0
1	1	0	1	0	0	0
1	1	1	0	0	0	0

2) (4 pts) Prove the following set equality properties by membership table:

$$A \cap (B \cap C) = (A \cap B) \cap C.$$

A	B	C	$B \cap C$	$A \cap B$	$A \cap (B \cap C)$	$(A \cap B) \cap C$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	1	0	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	0	1	0	0
1	1	1	1	1	1	1

3) (4 pts) Prove by simplification using set equality laws that

$$\left((A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B)\right) = (A \cup B).$$

1. $\left((A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B)\right)$ — *given*
2. $\left(\left(\left((A \cap B) \cup A\right) \cap \left((A \cap B) \cup \bar{B}\right)\right) \cup (\bar{A} \cap B)\right)$ — *Distributive Laws*
3. $\left(\left(\left(A \cup (A \cap B)\right) \cap \left(\bar{B} \cup (A \cap B)\right)\right) \cup (\bar{A} \cap B)\right)$ — *Commutative Laws*
4. $\left(\left(A \cap \left(\bar{B} \cup (A \cap B)\right)\right) \cup (\bar{A} \cap B)\right)$ — *Absorption Laws*
5. $\left(\left(A \cap \left((\bar{B} \cup A) \cap (\bar{B} \cup B)\right)\right) \cup (\bar{A} \cap B)\right)$ — *Distributive Laws*
6. $\left(\left(A \cap \left((\bar{B} \cup A) \cap U\right)\right) \cup (\bar{A} \cap B)\right)$ — *Inverse Laws*

7. $\left(\left(A \cap \left((\bar{B} \cup A)\right)\right) \cup (\bar{A} \cap B)\right)$ *Identity Laws*

8. $\left(\left(A \cap (A \cup \bar{B})\right) \cup (\bar{A} \cap B)\right)$ *Commutative Laws*

9. $\left(A \cup (\bar{A} \cap B)\right)$ *Absorption Laws*

10. $(A \cup \bar{A}) \cap (A \cup B)$ *Distributive Laws*

11. $U \cap (A \cup B)$ *Inverse Laws*

11. $(A \cup B)$ *Identity Laws*

4) (5 pts) Prove by simplification using set equality laws that

$$A \cap \left(\left((A \cup B) \cap C\right) \cup \left((A \cup B) \cap \bar{C}\right)\right) = A$$

$A \cap \left(\left(\left((A \cup B) \cap C\right) \cup (A \cup B)\right) \cap \left(\left((A \cup B) \cap C\right) \cup \bar{C}\right)\right)$ *Distributive laws*

$A \cap \left(\left((A \cup B) \cup \left((A \cup B) \cap C\right)\right) \cap \left(\bar{C} \cup \left((A \cup B) \cap C\right)\right)\right)$ *Commutative laws*

$A \cap \left((A \cup B) \cap \left(\bar{C} \cup \left((A \cup B) \cap C\right)\right)\right)$ *Absorption laws*

$A \cap \left((A \cup B) \cap \left(\left(\bar{C} \cup (A \cup B)\right) \cap (\bar{C} \cup C)\right)\right)$ *Distributive laws*

$A \cap \left((A \cup B) \cap \left(\left(\bar{C} \cup (A \cup B)\right) \cap U\right)\right)$ *Inverse laws*

$A \cap \left((A \cup B) \cap \left(\left(\bar{C} \cup (A \cup B)\right)\right)\right)$ *Identity laws*

$A \cap \left((A \cup B) \cap \left(\left((A \cup B) \cup \bar{C}\right)\right)\right)$ *Commutative laws*

$A \cap (A \cup B)$ *Absorption laws*

A *Absorption laws*

Please follow these directions for questions 5 - 8:

Prove the following subset and set equality properties by considering arbitrary elements of the subset and applying the principle of universal generalization. (For set equalities, you have to do this twice.)

5) (3 pts) $(A \cup B) \subseteq (A \cup B \cup C)$

$Let\ x\ be\ an\ element\ of\ (A \cup B), that\ means\ by\ definition\ of\ union\ that$
$x \in A\ \vee x \in B, since\ the\ statement\ x \in A\ \vee x \in B\ is\ true\ then$
$x \in A\ \vee x \in B \vee x \in C\ is\ true\ since\ for\ a\ statement\ of\ the\ form\ \ P \vee Q\ only\ P\ or\ Q$
$being\ true\ is\ enough\ for\ the\ whole\ statement\ to\ be\ true\ so\ if\ we\ treat$
$x \in A\ \vee x \in B\ as\ P\ and\ x \in C\ as\ Q, x \in A\ \vee x \in B\ being\ true\ is$
$enough\ for\ the\ whole\ statement\ of\ x \in A\ \vee x \in B \vee x \in C\ to\ be\ true. So$
$x \in A\ \vee x \in B \vee x \in C\ is\ true\ which\ means\ that\ x \in (A \cup B \cup C)\ is\ true, so$
$x \in (A \cup B) \rightarrow x \in (A \cup B \cup C), and\ since\ x\ was\ chosen\ arbitrarly\ we\ can\ use\ the$
$principle\ of\ universal\ generalization\ to\ say\ that\ for\ all\ elements\ y$
$if\ y\ is\ in\ (A \cup B)\ then\ y\ is\ in\ (A \cup B \cup C), or\ symbolically$
$\forall y\big(y \in (A \cup B) \rightarrow y \in (A \cup B \cup C)\big). and\ this\ is\ the\ same\ as\ saying\ that$
$(A \cup B) \subseteq (A \cup B \cup C)$

6) (3 pts) $(A \cap B \cap C) \subseteq (A \cap B)$

$Let\ x\ be\ an\ element\ of\ (A \cap B \cap C), that\ means\ by\ definition\ of\ intersection\ that$
$x \in A \wedge x \in B \wedge x \in C\ is\ true\ and\ \ since\ this\ is\ true, then\ x \in A \wedge x \in B\ is\ true,$
$since\ for\ x \in A \wedge x \in B \wedge x \in C\ to\ be\ true\ \ x \in A\ has\ to\ be\ true\ and\ x \in B\ has\ to\ be\ true$
$and\ x \in C\ has\ to\ be\ true\ and\ since\ x \in A\ and\ x \in B\ \ have\ to\ be\ true\ given\ that$
$x \in A \wedge x \in B \wedge x \in C\ is\ true, then\ x \in A \wedge\ x \in B\ is\ true\ which\ means$
$x \in (A \cap B), so\ x \in (A \cap B \cap C) \rightarrow x \in (A \cap B)\ \ and\ since\ x\ was\ chosen\ arbitrarily$
$we\ \ can\ use\ the\ principle\ of\ universal\ generalization\ to\ say\ that\ for\ all\ elements\ y$
$if\ y\ is\ in\ (A \cap B \cap C), then\ y\ is\ in\ (A \cap B),\ which\ symbolically\ means$
$\forall y\big(y \in (A \cap B \cap C) \rightarrow y \in (A \cap B)\big), which\ is\ the\ same\ as$
$saying\ (A \cap B \cap C) \subseteq (A \cap B).$

7) (6 pts) $(B - A) \cup (C - A) = (B \cup C) - A$

$Let\ x\ be\ an\ element\ of\ (B - A) \cup (C - A)\ that\ means\ by\ definition\ of\ set\ subtraction$
$and\ union\ that\ x\ is\ in\ B\ and\ not\ in\ A, or\ x\ is\ in\ C\ and\ not\ in\ A, which\ symbolically\ is$
$(x \in B \wedge x \notin A) \vee (x \in C \wedge x \notin A), we\ can\ represent\ the\ statements\ as\ symbols, we\ will$
$do\ thise\ like\ so, P\ will\ represent\ x \in B, Q\ will\ represent\ x \notin A\ and\ R\ will\ represent$
$x \in C. So, we\ have\ (P \wedge Q) \vee (R \wedge Q)\ and\ we\ can\ use\ derivations\ to\ try\ to\ derive$
$(P \vee R) \wedge Q\ which\ is\ (x \in B \vee x \in C) \wedge x \notin A.$

$(P \wedge Q) \vee (R \wedge Q)$ $\qquad\qquad$ $given$

$\big((P \wedge Q) \vee R\big) \wedge \big((P \wedge Q) \vee Q\big)$ *distributive laws*
$\big((P \wedge Q) \vee R\big) \wedge Q$ *absorption laws*
$\big((P \vee R) \wedge (Q \vee R)\big) \wedge Q$ *distributive laws*
$(P \vee R) \wedge \big((Q \vee R) \wedge Q\big)$ *associative laws*
$(P \vee R) \wedge Q$ *absorption laws*

So we have derived $(P \vee R) \wedge Q$ *so we can say that* $(P \wedge Q) \vee (R \wedge Q) \rightarrow (P \vee R) \wedge Q$, (*note that deriving via just logic laws you can say that* $(P \wedge Q) \vee (R \wedge Q) \leftrightarrow (P \vee R) \wedge Q$ *and that means* $(P \wedge Q) \vee (R \wedge Q) \rightarrow (P \vee R) \wedge Q$ *and* $(P \vee R) \wedge Q \rightarrow (P \wedge Q) \vee (R \wedge Q)$ *are both true*).
So, we can say that $(x \in B \wedge x \notin A) \vee (x \in C \wedge x \notin A) \rightarrow (x \in B \vee x \in C) \wedge x \notin A$, *and* $(x \in B \vee x \in C) \wedge x \notin A$ *is the same as saying that* $x \in (B \cup C) - A$, *so we can say that* $x \in (B - A) \cup (C - A) \rightarrow x \in (B \cup C) - A$ *and by since x was chosen arbitrarily we can use the principle of universal generalization to say that for all elements y if y is in* $(B - A) \cup (C - A)$ *then y is in* $(B \cup C) - A$, *or in symbolic notation we can say* $\forall y(y \in (B - A) \cup (C - A) \rightarrow y \in (B \cup C) - A)$, *which is the same as saying that* $(B - A) \cup (C - A) \subseteq (B \cup C) - A$. *Now we must show that* $(B \cup C) - A \subseteq (B - A) \cup (C - A)$, *to show that* $(B - A) \cup (C - A) = (B \cup C) - A$.

So now we will choose an arbitrary element z in $(B \cup C) - A$, *that means that* $(z \in B \vee z \in C) \wedge z \notin A$, *we can represent* $z \in B$ *as T*, $z \in C$ *as S and* $z \notin A$ *as K, so we can write* $(z \in B \vee z \in C) \wedge z \notin A$ *as* $(T \vee S) \wedge K$, *and since we already proved that* $(P \wedge Q) \vee (R \wedge Q) \leftrightarrow (P \vee R) \wedge Q$, *and P, Q and R are arbitrary sentences we can say that* $(T \vee S) \wedge K \leftrightarrow (T \wedge K) \vee (S \wedge K)$ *and* $(T \vee S) \wedge K \leftrightarrow (T \wedge K) \vee (S \wedge K)$ *implies that* $(T \vee S) \wedge K \rightarrow (T \wedge K) \vee (S \wedge K)$ *is true and* $(T \wedge K) \vee (S \wedge K) \rightarrow (T \vee S) \wedge K$ *is true*
So we can say $(T \vee S) \wedge K \rightarrow (T \wedge K) \vee (S \wedge K)$ *which if we do substitution again is* $(z \in B \vee z \in C) \wedge z \notin A \rightarrow (z \in B \wedge z \notin A) \vee (z \in C \wedge z \notin A)$ *and since* $(z \in B \wedge z \notin A) \vee (z \in C \wedge z \notin A)$ *is the same as saying* $z \in (B - A) \cup (C - A)$, *we can say* $z \in (B \cup C) - A \rightarrow z \in (B - A) \cup (C - A)$ *and since z was chosen arbitrarly, we can use the principle of universal generalization, to say that for all elements y if y is in* $(B \cup C) - A$ *then y is in* $(B - A) \cup (C - A)$, *which in symbolic notation is* $\forall y\big(y \in (B \cup C) - A \rightarrow y \in (B - A) \cup (C - A)\big)$ *and is the same as saying* $(B \cup C) - A \subseteq (B - A) \cup (C - A)$. *So we have* $(B \cup C) - A \subseteq (B - A) \cup (C - A)$ *and* $(B - A) \cup (C - A) \subseteq (B \cup C) - A$ *since both are subsets of each other then by definition* $(B - A) \cup (C - A) = (B \cup C) - A$.

8) (6 pts) $A \cup (A \cap B) = A$

Let x be an element of $A \cup (A \cap B)$, *that means by definition of union and intersection that* $x \in A \vee (x \in A \wedge x \in B)$, *we can now represent* $x \in A$ *as P and* $x \in B$ *as Q, so we can write* $x \in A \vee (x \in A \wedge x \in B)$ *as* $P \vee (P \wedge Q)$, *now we will prove that* $P \vee (P \wedge Q)$ *can be written as P, meaning we will prove* $P \vee (P \wedge Q) \leftrightarrow P$.

$P \vee (P \wedge Q)$	$given$
P	$absorption\ laws$

$Thus\ P \vee (P \wedge Q) \leftrightarrow P, meaning\ P \vee (P \wedge Q) \rightarrow P\ and\ P \rightarrow P \vee (P \wedge Q), so\ by\ substituting,$ $we\ get\ x \in A \vee (x \in A \wedge x \in B) \rightarrow x \in A, and\ thus\ we\ can\ say\ that$ $x \in A \cup (A \cap B) \rightarrow x \in A\ and\ since\ we\ used\ an\ arbitrary\ x\ we\ can\ use\ the\ principle$ $of\ universal\ generalization\ to\ say\ that\ for\ all\ elements\ y\ if\ y\ is\ in\ A \cup (A \cap B), then$ $y\ is\ in\ A, or\ symbollicaly\ \forall y(y \in A \cup (A \cap B) \rightarrow y \in A)\ which\ is\ the\ same\ as\ saying\ that$ $A \cup (A \cap B) \subseteq A. Now\ we\ need\ to\ prove\ that\ A \subseteq A \cup (A \cap B), to\ prove\ that$ $A \cup (A \cap B) = A.$

$Now\ let\ z\ be\ an\ element\ of\ A, that\ means\ z \in A\ and\ this\ can\ be\ written\ as\ R, like$ $before\ P \vee (P \wedge Q) \leftrightarrow P\ and\ since\ P\ and\ Q\ are\ arbitrary\ sentences, it\ can\ be\ said\ that$ $R \vee (R \wedge S) \leftrightarrow R, where\ S\ is\ z \in B, although\ it\ could\ be\ anything. So, it\ can\ be\ said\ that$ $R \rightarrow R \vee (R \wedge S), and\ thus\ z \in A \rightarrow\ z \in A \cup (A \cap B)\ and\ since\ z\ was\ an\ arbitrary$ $element\ we\ can\ use\ the\ principle\ of\ universal\ generalization\ to\ say\ that\ for\ all\ elements$ $y, if\ y\ is\ in\ A, then\ y\ is\ in\ A \cup (A \cap B), or\ in\ symbolic\ notation, it\ would\ be$ $\forall y\big(y \in A \rightarrow y \in A \cup (A \cap B)\big), which\ is\ the\ same\ as\ saying\ A \subseteq A \cup (A \cap B), and\ thus$ $since\ both\ A \subseteq A \cup (A \cap B)\ and\ A \cup (A \cap B) \subseteq A\ are\ true, A \cup (A \cap B) = A\ is\ true.$

9) (5 pts) Prove or disprove for arbitrary sets *A*, *B* and *C*:

$$\text{if } (B - A) \subseteq (C - A), \text{ then } B \subseteq C.$$

This is false, let’s use a counter example to prove it is false.

$B = \{1, 2, 3\}, A = \{1, 2\}, C = \{2, 3, 4\}$
$B - A = \{3\}, \quad C - A = \{3, 4\}, \quad \{3\} \subseteq \{3, 4\}, \ so\ (B - A) \subseteq (C - A)$
$\{1, 2, 3\} \nsubseteq \{2, 3, 4\}, and\ thus\ B \nsubseteq C.$
$So, we\ have\ an\ example\ of\ (B - A)\ being\ the\ subset\ of\ (C - A)\ and\ at\ the\ same\ time$ $B\ not\ being\ a\ subset\ of\ C\ and\ thus\ we\ have\ disproven\ the\ original\ statement.$

10) (5 pts) Prove or disprove for arbitrary sets *A*, *B* and *C*:

$$\text{if } B \subseteq C, \text{ then } A \cap B \subseteq A \cap C$$

$So, we\ will\ prove\ this\ statement\ by\ showing\ that\ if\ we\ assume\ that\ B \subseteq C\ is\ true, it$ $will\ be\ the\ case\ that\ A \cap B \subseteq A \cap C\ is\ true. Let'assume\ B \subseteq C\ is\ true, that\ means\ that$ $for\ all\ elements\ y\ if\ y\ is\ in\ B, then\ y\ is\ in\ C, so\ \forall y(y \in B \rightarrow y \in C), now\ we\ want\ to$ $prove\ that\ \forall y(y \in A \cap B \rightarrow y \in A \cap C)\ which\ is\ the\ same\ as\ saying\ that\ A \cap B \subseteq A \cap C$ $so\ given\ \forall y(y \in B \rightarrow y \in C), we\ want\ to\ show\ that\ for\ all\ elements\ y\ if\ y\ is\ in\ A \cap B,$ $then\ y\ will\ be\ in\ A \cap C. So, lets\ assume\ that\ for\ some\ arbitrary\ element\ x, x\ is\ in$ $A \cap B, meaning\ x \in A \wedge x \in B, since\ this\ is\ a\ conjunction\ both\ x \in A\ and\ x \in B\ are\ true.$

$Since\ \forall y(y \in B \rightarrow y \in C), we\ can\ use\ the\ principle\ of\ universal\ specification\ to\ say\ that\ x \in B \rightarrow x \in C\ and\ since\ x \in B\ since\ x \in B\ and\ x \in A, we\ can\ say\ that\ x \in C, so\ we\ have\ that\ x \in C\ and\ x \in A\ and\ thus\ we\ can\ say\ that\ x \in A \wedge x \in C, which\ is\ the\ same\ as\ saying\ that\ x \in A \cap C, thus\ by\ assuming\ \ x \in A \cap B, we\ derived\ x \in A \cap C, thus\ we\ can\ say\ that\ x \in A \cap B \rightarrow x \in A \cap C, and\ since\ x\ is\ an\ arbitrary\ element\ we\ can\ use\ the\ principle\ of\ universal\ generalization\ to\ say\ that\ for\ all\ elements\ y\ if\ \ y\ is\ in\ A \cap B, then\ y\ is\ in\ A \cap C, which\ symbolically\ can\ be\ written\ as\ the\ following\ \ \forall y(y \in A \cap B \rightarrow y \in A \cap C), and\ thus\ we\ have\ proven\ that\ if\ B \subseteq C\ then\ A \cap B \subseteq A \cap C.$

11) (5 pts) Give a summary of the life and mathematical contributions of Daniel Bernoulli. Make sure to mention the Bernoulli Principle.

Sample Summary

The Bernoulli family is one of the best known families in mathematics, with multiple family members achieving notoriety in their lifetimes. Johann Bernoulli, Daniel's father as well as Jacob Bernoulli, Johann's brother, grew up in the late 1600s, just as the study of Calculus was being born. Both Jacob and Johann took an interest in mathematics, and Calculus, specifically, and are some of the first mathematicians to apply Calculus to solving a variety of problems.

Daniel was born in near the turn of the eighteenth century, on February 8, 1700 in the Netherlands. He was Johann's second son and Johann tutored Daniel in mathematics. Incidentally, Johann also tutored perhaps the greatest mathematician of all time, Leonard Euler, and Daniel and Leonard were mathematical contemporaries. Just like his father, Daniel initially studied medicine in university, but actually had a passion for mathematics. Early in his career, Daniel wrote a well-regarded paper on fluid flow, which helped him earn a faculty post at the prestigious Academy of Sciences at St. Petersburg. He happened to be there the same time Euler was and the two frequently worked together. He stayed at his post in St. Petersburg, where he lectured on medicine, physics and mechanics for eight years before returning to his native Switzerland, taking a post at the University of Basel, in 1732.

Interestingly enough, Daniel had quite the feud with his father Johann. Johann at times seemed jealous that Daniel had superior mathematical talent to himself. For example, in a paper they co-authored in 1735 on planetary orbits, Johann was extremely upset that Daniel got any credit for the work at all. So upset, in fact, that he supposedly threw Daniel out of the house! Later, in 1738, Daniel published one of his most famous works, Hydrodynamica, which further explored fluid flow, relating pressure, density and velocity. It is from this work that the famed Bernoulli Principle was published, namely that pressure in a fluid decreases as its velocity increases. Bernoulli came up with the idea of puncturing a pipe with a small straw and seeing how far up the straw the fluid went. Doctors actually used this idea for measuring blood pressure (painful!) for close to 200 years before a better method was discovered.

Daniel lived until 1782, and continued his post at the University of Basel throughout the rest of his career. Over this time he won ten Grand Prizes from the Academy of Paris for work dealing with economics, navigation and astronomy.

Sources

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