Fall 2020 COT 3100 Homework #1 Solution

1) (5 pts) Multiply $(x^4 + 5x^3 + 2x) \times (3x^2 - 4x + 7)$. What is the fully expanded result?

Solution

$(x^4 + 5x^3 + 2x) \times (3x^2 - 4x + 7) =$

$(3x^6 - 4x^5 + 7x^4) + (15x^5 - 20x^4 + 35x^3) + (6x^3 - 8x^2 + 14x) =$

$\mathbf{3x^6 + 11x^5 - 13x^4 + 41x^3 - 8x^2 + 14x}$

2) (4 pts) Create a truth table for the following logical expression: $(p \wedge q) \vee \neg r$. Please make sure your first three columns are for p, q and r, respectively, and that you have appropriate intermediary columns. In addition, make sure that the rows are ordered in numerical order, in binary, if we treat False as the bit 0 and True as the bit 1.

Solution

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\neg \boldsymbol{r}$	$\boldsymbol{p} \wedge \boldsymbol{q}$	$(\boldsymbol{p} \wedge \boldsymbol{q}) \vee \neg \boldsymbol{r}$
F	F	F	T	F	T
F	F	T	F	F	F
F	T	F	T	F	T
F	T	T	F	F	F
T	F	F	T	F	T
T	F	T	F	F	F
T	T	F	T	T	T
T	T	T	F	T	T

3) (8 pts) Create a truth table for the following logical expression: $\neg(p \vee q) \wedge (r \oplus s)$. Please make sure your first three columns are for p, q and r, respectively, and that you have appropriate intermediary columns. In addition, make sure that the rows are ordered in numerical order, in binary, if we treat False as the bit 0 and True as the bit 1.

Solution

p	q	r	s	$p \vee q$	$\neg(p \vee q)$	$r \oplus s$	$\neg(p \vee q) \wedge (r \oplus s)$
F	F	F	F	F	T	F	F
F	F	F	T	F	T	T	T
F	F	T	F	F	T	T	T
F	F	T	T	F	T	F	F
F	T	F	F	T	F	F	F
F	T	F	T	T	F	T	F
F	T	T	F	T	F	T	F
F	T	T	T	T	F	F	F
T	F	F	F	T	F	F	F
T	F	F	T	T	F	T	F
T	F	T	F	T	F	T	F
T	F	T	T	T	F	F	F
T	T	F	F	T	F	F	F
T	T	F	T	T	F	T	F
T	T	T	F	T	F	T	F
T	T	T	T	T	F	F	F

4) (6 pts) If it's raining, the ground outside is wet. However, if the ground outside is wet, that does not necessarily mean it's raining. So, we can say that the fact it's raining *implies* that the ground outside is wet – but not vice versa. **Give at least three** real-life examples of similar situations where, in formal terms, $p \to q$ but $\neg(q \to p)$: where a condition p necessarily implies a condition q, but the reverse is not true.

- For each example, **explain** why q must be true if p is true.
- For each example, **give a situation** where q can be true without p being true.

Solution

Many examples work. Here are a three:

1. If person X is an active UCF student, then person X attends a university. (However, if person A attends a university, they might not attend UCF, like those unfortunate students in Gainesville =))

2. If person X earns an A in COT 3100, then they understand several discrete mathematics concepts. (The various requirements of the course to earn an A aren't possible without an understanding of several discrete mathematics topics. Unfortunately, one who understands these topics may not earn an A in the course for various reasons (forgot to take an exam, didn't explain

their logic properly in answering homework and exam questions, were preoccupied by a personal matter and performed poorly on a test even though they understood the concepts).

3. If person X wins the Powerball Lottery, then person X will be a millionaire. (This is true since the payouts for the Powerball are quite large. However, if someone is a millionaire, they may have attained that status in a different way, perhaps by starting their own business or becoming a doctor!)

The common thread in these examples is that the if clause is more restrictive than the then clause. Namely, there is more than one way to achieve the then clause and the given if clause is just one of those possible ways.

5) (6 pts) Use the truth table method to prove the following two expressions are logically equivalent:

$$(p \lor q) \to r$$

$$(p \to r) \land (q \to r)$$

Solution

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{p \lor q}$	$\boldsymbol{p \to r}$	$\boldsymbol{q \to r}$	$\boldsymbol{(p \lor q) \to r}$	$\boldsymbol{(p \to r) \land (q \to r)}$
F	F	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	T	F	T	T	F	F	F
F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F
T	F	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	T	T	T	T	T	T	T

Since the last two columns are identical, it follows that the two given expressions are equivalent, since we've evaluated them for each possible truth setting of the variables p, q and r.

6) (6 pts) Use the laws of logic to show that two following expressions are logically equivalent:

$$(p \wedge (\bar{p} \rightarrow r)) \vee ((q \wedge s) \vee (q \wedge \bar{s}))$$

$$p \vee q$$

Solution

$\big(p \wedge (\bar{p} \rightarrow r)\big) \vee \big((q \wedge s) \vee (q \wedge \bar{s})\big) \leftrightarrow$	
$\big(p \wedge (\bar{p} \rightarrow r)\big) \vee (q \wedge (s \vee \bar{s})) \leftrightarrow$	Distributive Law
$\big(p \wedge (\bar{p} \rightarrow r)\big) \vee (q \wedge T) \leftrightarrow$	Inverse Law
$\big(p \wedge (\bar{p} \rightarrow r)\big) \vee q \leftrightarrow$	Identity Law
$\big(p \wedge (\bar{\bar{p}} \vee r)\big) \vee q \leftrightarrow$	Implication Identity
$\big(p \wedge (p \vee r)\big) \vee q \leftrightarrow$	Double Negation
$p \vee q$	Absorption Law

7) (5 pts) Use the rules of inference to prove the following argument:

$p \vee q$
$p \rightarrow r$
$q \rightarrow r$
$s \vee t$
$\bar{t}$

$\therefore r \wedge s$

Solution

1. $p \rightarrow r$	Premise
2. $q \rightarrow r$	Premise
3. $(p \vee q) \rightarrow r$	Proof by Cases with 1, 2
4. $p \vee q$	Premise
5. r	Law of Syllogism with 4, 3
6. $s \vee t$	Premise
7. $\bar{t}$	Premise
8. s	Rule of Disjunctive Syllogism with 6, 7
9. $r \wedge s$	Rule of Conjunction with 5, 8

8) (5 pts) Prove or disprove the following claim over the universe of all real numbers for x and y.

$$\forall x\left[\exists y[x^2 - 5xy + 6y^2 = 0]\right]$$

If the claim is false, find a value of x for which it is false. If it is true, show which value(s) of exist to make the claim true.

Solution

The claim is true. Start with the given equation and factor

$$x^2 - 5xy + 6y^2 = 0$$
$$(x - 2y)(x - 3y) = 0$$

It follows that if $y = x/2$ or $y = x/3$, that the given expression will be 0 and the given statement will be true. Thus, there are usually 2 values for any x that make the statement true, namely $y = x/2$ and $y = x/3$. Note that when x = 0, there is only one value of y, y = 0 that makes the statement true.

9) (5 pts) The last question of each homework assignment will be to write up a two paragraph summary of a topic from the history of mathematics. The idea here is that rarely is any of this history taught in mathematics classes and while I don't have class time to teach it, I thought it would be nice if students learned a bit for each homework assignment. There's no need to use fancy sources, websites will do, but please site which websites you pulled your information from.

Give a summary of the life and mathematical contributions of Sophie Germain. If you are so inclined, for fun, write a program that prints out all Sophie Germain primes less than 1,000,000. Include your source code in your .pdf submission. (Please submit C, Java or Python code.) Note, no points are assigned for the program.

Sample Summary

Sophie Germain was born into a well to do French family in 1776. As young as thirteen, Sophie began reading mathematical classics in her father's library, eventually reading works by Newton and Euler. She showed a particular interest in number theory. In spite of the fact that her parents actively dissuaded her from studying mathematics, she stubbornly pursued it. In 1794, even though she wasn't allowed to attend the Ecole Polytechnique (an university), she took advantage of the fact that the school made lecture notes available to anyone who requested them. Reading these notes eventually led to correspondence with one of the faculty at the school, Joseph Lagrange. Afraid that he wouldn't approve of her being a woman, she corresponded with him under a male pseudonym. Lagrange was impressed with Germain's work and requested a meeting, at which point she had to make her real identity known to him. Luckily, he didn't mind that she was a she, and he became a mentor to her.

Germain is known for both her work in number theory and elasticity. Early in her career, she corresponded with both Legendre and Gauss. In her correspondence with the latter, she attempted to attack portions of Fermat's Last Theorem. Though Gauss was sometimes reluctant with his correspondence with Sophie, he was genuinely impressed with her work and publicly praised her mathematical achievements. He was instrumental in the University of Gottingen posthumously awarding her an honorary degree.

The Paris Academy of Sciences posted an award for anyone who could come up with a mathematical theory to describe the vibration of an elastic surface. The problem was so difficult that there were ultimately only two entrants: Germain and Denis Poisson. But when Poisson was elected to the Academy, he was no longer allowed to be a participant, leaving only Germain. It took Germain three attempts in 1811, 1813 and 1816 before she was finally awarded the prize. She did consult with both Lagrange and Poisson during her work and she continued to publish papers in elasticity until her death.

After 1816, Germain returned to focus on number theory and Fermat's Last Theorem. Using the theorem named after her, she was able to prove Fermat's Last Theorem for every odd prime less than 100. In her theorem, she refers to odd primes p such that there exists an auxiliary prime $P = 2Np + 1$, for some integer N not divisible by 3. In a special case of this definition, where $N = 1$, prime numbers p such that $2p+1$ are also prime are known as Sophie Germain primes. The first four Sophie Germain primes are 2, 3, 5 and 11.

In 1829, Germain found out that she had breast cancer. She continued to work on her mathematics at this time, passing away at the age of 55.

A separate attachment (sophie.java) prints out all of the Sophie Germain primes up to 1,000,000. The program works by running a prime sieve to 2,000,000 and checking for each prime p, up to 1,000,000, if 2p+1 is also prime or not. The output of the program was piped to the file sophie1000000.txt.

Source: https://en.wikipedia.org/wiki/Sophie_Germain