COT 3100 Section 2 Final Exam - Part D (Counting, Probability) - 35 pts Solutions

1) (5 pts) A subsequence of a sequence is any sequence that can be obtained from the original sequence by deleting 0 or more of the items and keeping all the items in the same order. For example, the sequence A, B, C, D, E, F has the subsequence B, C, F, obtained by deleting A, D and E. How many non-empty subsequences are there of the sequence Q, U, A, K, E, R?

Solution

Each subset of the letters corresponds to a subsequence and since each of the letters are distinct, each subsequence is distinct. It follows that there are **$2^6 - 1 = 63$** non-empty subsequences (the total number of subsets of 6 items minus the empty set.)

Grading: Mostly all or nothing. 5 pts for 63 as long as the reason is given 4 pts for 64 (take off 1 for forgetting the -1). It's hard to see other answers that are worth any credit, bit if you see something, feel free to award a bit of partial credit.

2) (10 pts) Jabba the Hut lives in New York City and loves to eat. We can model New York City as the Cartesian grid. Naturally, each street corner (intersection of a North-South Avenue and East-West Street) is a lattice point on the grid. Jabba lives at (0, 0) and goes to work every morning, which is located at (12, 10), which is 12 blocks east and 10 blocks north of his home. He walks 22 blocks each day to work. (Walking a block means starting at the lattice point (x, y) and either walking east one block to (x+1, y) or north one block to (x, y+1).) Each street corner, except Jabba's home, (0, 0), has a hot dog stand and a pretzel stand. If Jabba arrives at a corner by walking north, he always buys a pretzel at the corner when he arrives, and if he arrives at the corner by walking east, he always buys a hot dog. Jabba has identified that the hot dog stand at (4, 3) serves bad hot dogs. He has also identified that both the pretzel stand AND hot dog stand at (7, 8) serve bad pretzels and hot dogs, respectively. He is determined to avoid all three of these stands while still getting his requisite 22 treats on his way to work. How many different paths can Jabba take to work?

Solution

First, we know that all of Jabba's possible walking paths are strings of length 22 consisting of 12 Es and 10 Ns, where an E represents walking east and an N represents walking N. We can use the information in the question to count only the strings that adhere to the given restrictions from this total set of $\binom{22}{10}$ strings.

First, let's just count the total number of paths that go through point (7, 8), since we know that we simply can't visit this point, since both stands are to be avoided at it.

We can get to (7, 8) in $\binom{15}{7}$ ways, since we must choose 7 of the first 15 moves to go east. We can go from (7, 8) to (12, 10) in $\binom{7}{5}$ ways, since we must choose 5 of the last 7 moves to go east. Thus, there are $\binom{15}{7}\binom{7}{5}$ paths that go through the forbidden street corner.

Now, to let's count the number of paths that arrive at (4, 3) by moving east on the last step before getting to (4, 3): This means that the first 6 letters must be 3 N's and 3 E's, which we can arrange in $\binom{6}{3}$ ways. The 7th letter must be an E. The following 15 letters must be 8 Es and 7 Ns, which can be arranged in $\binom{15}{7}$ ways as well. Thus, there are $\binom{6}{3}\binom{15}{7}$ paths where Jabba gets a bad hot dog at (4, 3).

Finally, there are going to be some paths where Jabba gets a bad hot dog at (4, 3) AND visits the point (7, 8). Let's count how many of these there are:

There are $\binom{6}{3}$ paths which get to (4, 3), moving east on the seventh step. From there, there are $\binom{8}{3}$ paths that move from (4, 3) to (7, 8), since here we have to choose 3 times to go east out of the next 8 moves to get to (7, 8). Finally, from (7, 8), there are $\binom{7}{5}$ to get to (12, 10). It follows that there are $\binom{6}{3}\binom{8}{3}\binom{7}{5}$ paths that both go through (4, 3) moving north to arrive at the point AND go through (7, 8).

Using the Inclusion-Exclusion Principle, it follows that the total number of "good ways" Jabba can get to work is:

$$\binom{22}{10} - \binom{15}{7}\binom{7}{5} - \binom{6}{3}\binom{15}{7} + \binom{6}{3}\binom{8}{3}\binom{7}{5}$$

This is sufficiently complicated so it's best to leave the final answer in this form. (It's unlikely that there will be any dramatic simplification of these terms.) Incidentally, this simplifies to 406,331.

Grading: 3 pts for total (22 choose 10)
3 pts to sub out paths through (7, 8)
3 pts to sub out paths through (4, 3) that move east last
1 pt to adjust for I/E

3) (5 pts) A bag contains 6 blue marbles, 7 green marbles and 8 red marbles. If three marbles are chosen out of the bag (without replacement) without looking, what is the probability that all three marbles are blue? Express your answer as a fraction in lowest terms.

Solution

We can choose the three marbles out of 21 in $\binom{21}{3}$ ways. Of these choices, there are $\binom{6}{3}$ ways to choose only blue marbles. Our desired probability is $\frac{\binom{6}{3}}{\binom{21}{3}} = \frac{6\times5\times4}{21\times20\times19} = \frac{2}{133}$.

Grading: 2 pts for denominator, 2 pts for numerator, 1 pt to get to fraction in lowest terms.

4) (10 pts) There are n Christmas lights in a row. Three of these n lights are randomly selected and turned on. The probability that no two of the lights turned on are next to each other is $\frac{39}{60}$. What is n? (Note: You should get a rather ugly quadratic equation when you work this out. Please show all the work to arrive at the quadratic, but use your calculator to find the roots of it.)

Solution

The sample space is $\binom{n}{3}$, since we can choose 3 lights out of n in that many ways.

Of these choices, we can calculate how many don't have consecutive lights by using the n-3 lights turned off as separators:

__ O __ O __ O … __ O __

The lights that are turned on can be placed in the blanks (There are n-2 of these blanks since there are n-3 Os.). We can place at most one light per blank, so we can do this in $\binom{n-2}{3}$ ways.

Thus the desired probability is $\frac{\binom{n-2}{3}}{\binom{n}{3}} = \frac{(n-2)(n-3)(n-4)}{n(n-1)(n-2)} = \frac{(n-3)(n-4)}{n(n-1)}$

Now, set this equal to $\frac{39}{60}$.

$$\frac{(n-3)(n-4)}{n(n-1)} = \frac{39}{60}$$

$$60(n-3)(n-4) = 39n(n-1)$$

$$60n^2 - 420n + 720 = 39n^2 - 39n$$

$$21n^2 - 381n + 720 = 0$$

$$7n^2 - 127n + 240 = 0$$

$$(7n-15)(n-16) = 0$$

Since n is an integer, the desired answer is **n = 16.**

Grading: 2 pts for sample space, 4 pts for numerator, 3 pts to get to quadratic, 1 pt answer

5) (5 pts) In what state did the chain restaurant California Pizza Kitchen open its first store?

California, give to all who submit something for the section.