COT 3100 Section 2 Final Exam - Part C (Number Theory, Induction) - 30 pts Solutions

1) (5 pts) Let $n = 2^9k$, where k is a positive odd integer. What percentage of n's divisors are even?

Solution

Let k have d divisors. Then n has 10d divisors, since any divisor of k can be multiplied by 2^0 or 2^1 or 2^2, or …, 2^9 to create all the divisors of n. Of this list of divisors, only the ones which are of the form $2^0d'$, where d' is a divisor of k, are odd. This is precisely d of those divisors. It follows that 9d divisors of n are even, and 9d out of a total of 10d is **90% of the divisors.**

Grading: 2 pts for stating in some way that there are 10 possible exponents to 2 in the divisors of n, 2 pts for observing that each of these is equally likely in a randomly chosen divisor of n, 1 pt for concluding that 90% is the answer from these observations.

2) (5 pts) Let gcd(a, b) denote the greatest common divisor of positive integers a and b and let lcm(a, b) denote the least common multiple of positive integers a and b. Let a, b and c be arbitrary positive integers. Simplify the following expression so that your final answer is solely in terms of a, b and c, and does not contain any reference to the gcd or lcm of pairs of integers:

$$\frac{\gcd(a,b)\gcd(b,c)lcm(a,b)lcm(b,c)}{\gcd(a,c)lcm(a,c)}$$

Solution

Recall in class that we proved for any positive integers a and b, that gcd(a, b)lcm(a, b) = ab, use this formula and substitute:

$$\frac{\gcd(a,b)\gcd(b,c)\,lcm(a,b)lcm(b,c)}{\gcd(a,c)\,lcm(a,c)} = \frac{\gcd(a,b)\,\mathrm{lcm}(a,b)\,gcd(b,c)lcm(b,c)}{\gcd(a,c)\,lcm(a,c)} = \frac{(ab)(bc)}{(ac)} = \boldsymbol{b^2}$$

Grading: 2 pts for stating the fact about gcd, lcm, 3 pts for substituting that fact and simplifying.

3) (10 pts) Let F_n denote the n^{th} Fibonacci number. (Recall $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all integers $n > 2$.) Using induction on n, prove the following summation formula for all positive integers n:

$$\sum_{i=1}^{2n-1} F_i F_{i+1} = F_{2n}^2$$

Solution

Base case: n = 1, LHS = $\sum_{i=1}^{2(1)-1} F_i F_{i+1} = F_1 F_2 = 1 \times 1 = 1$

RHS = $F_{2(1)}^2 = F_2^2 = 1^2 = 1$

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{2k-1} F_i F_{i+1} = F_{2k}^2$$

Inductive step: Prove for n = k + 1 that

$$\sum_{i=1}^{2(k+1)-1} F_i F_{i+1} = F_{2(k+1)}^2$$

$\sum_{i=1}^{2(k+1)-1} F_i F_{i+1} = \sum_{i=1}^{2k+1} F_i F_{i+1}$

$= \left(\sum_{i=1}^{2k-1} F_i F_{i+1}\right) + F_{2k} F_{2k+1} + F_{2k+1} F_{2k+2}$, splitting the sum

$= F_{2k}^2 + F_{2k} F_{2k+1} + F_{2k+1} F_{2k+2}$, using the IH

$= F_{2k}(F_{2k} + F_{2k+1}) + F_{2k+1} F_{2k+2}$, factoring F_{2k} from first 2 terms

$= F_{2k}(F_{2k+2}) + F_{2k+1} F_{2k+2}$, using the Fibonacci recurrence

$= F_{2k+2}(F_{2k} + F_{2k+1})$, factoring F_{2k+2} from the 2 terms

$= F_{2k+2}(F_{2k+2})$, using the Fibonacci recurrence

$= F_{2k+2}^2$, definition of square…

Grading: 1 pt base case, 1 pt IH, 2 pts IS, 1 pt split sum, 2 pts use IH, 3 pts rest of the algebra, give partial as you see fit.

4) (10 pts) Let *a* be an odd integer. Prove for all positive integers *n* via induction that $a^{2^n} - 1$ is divisible by 2^{n+2}.

Solution

Base case: n = 1, we must show that $a^{2^1} - 1 = a^2 - 1 = (a-1)(a+1)$ is divisible by 2^{1+2} = 8.

Since a is odd, there exists an integer z such that a = 2z + 1. Substitute to get

$$(a-1)(a+1) = (2z+1-1)(2z+1+1) = (2z)(2z+2) = 4z(z+1)$$

Now, note that z must be either even of odd. If z is even z(z+1) is even. If z is odd, then z+1 is even and it also follows that z(z+1) is even. Thus, we can conclude that z(z+1) is even. Thus, for some integer y, z(z+1) = 2y, substituting we get:

$$4z(z+1) = 4(2y) = 8y = 2^3 y$$

This proves that the desired quantity is divisible by 8, proving that the given statement is true for the base case, n = 1.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that $a^{2^k} - 1$ is divisible by 2^{k+2}.

Inductive step: prove for n = k + 1 that $a^{2^{k+1}} - 1$ is divisible by 2^{k+3}.

$a^{2^{k+1}} - 1 = (a^{2^k} + 1)(a^{2^k} - 1)$, using the difference of squares factoring formula

Now, utilizing the inductive hypothesis, $2^{k+2} | (a^{2^k} - 1)$. It follows that there exists an integer c such that $a^{2^k} - 1 = c(2^{k+2})$. Substitute accordingly:

$$= (a^{2^k} + 1)c(2^{k+2})$$

Since a is odd, any positive integer power of a is odd as well, thus a^{2^k} is odd and $a^{2^k} + 1$ is even. Thus, by definition of even numbers, there exists some integer d such that $a^{2^k} + 1 = 2d$. Substitute:

$$= (2d)c(2^{k+2})$$
$$= (2^{k+3})(cd)$$

Since c and d are integers and we've expressed the quantity in question as a multiple of 2^{k+3}, it follows that $a^{2^{k+1}} - 1$ is divisible by 2^{k+3}, as desired, proving the inductive step.

We can conclude that the given statement is true for all positive integers n.

Grading: 5 pts base case, 1 pt IH, 1 pt IS, 1 pt factor, 1 pt use IH, 1 pt stating last term is odd and concluding