

COT 3100 Section 2 Final Exam - Part B (Logic, Sets, Relations, Functions) - 30 pts
Solutions

1) (8 pts) Use a **truth table** to show that the Rule of Resolution on the Rules of Inference list is a one way implication. Namely, prove that the logical expressions to the left and right of the implication sign are **NOT** equivalent. Your table should have seven columns.

Solution

p	q	r	$p \vee q$	$\bar{p} \vee r$	$(p \vee q) \wedge (\bar{p} \vee r)$	$q \vee r$
F	F	F	F	T	F	F
F	F	T	F	T	F	T
F	T	F	T	T	T	T
F	T	T	T	T	T	T
T	F	F	T	F	F	F
T	F	T	T	T	T	T
T	T	F	T	F	F	T
T	T	T	T	T	T	T

Grading: 1 point per row, the row has to be all correct to get the point. If a column is missing, take off 1 pt total for the column being missing.

2) (7 pts) Let A , B and C be arbitrary sets. **Without the aid of a membership table**, prove:

$$\text{if } A - B = C - B, \text{ then } A \cup B = C \cup B$$

Note: The proof is long since you have to show that both the LHS is a subset of the RHS and that the RHS is a subset of the LHS. But, the second part of the proof is symmetric to the first (since swapping the roles of A and C doesn't change the problem at all). Thus, just prove the first part and note that the symmetry proves the other subset relation.

Solution

We prove the equality of sets by showing that the subset relationship holds in both directions, namely, we must prove that:

- (a) *if $A - B = C - B$, then $A \cup B \subseteq C \cup B$* , and
- (b) *if $A - B = C - B$, then $C \cup B \subseteq A \cup B$*

Notice that both statements are symmetric, since the roles of A and C can be switched in the first to produce the second statement. Thus, if we can prove part (a), that serves as a proof of part (b) and we can conclude that the sets $A \cup B$ and $C \cup B$ are equal, as desired.

To prove that one set is a subset of another, we must show that an arbitrarily chosen element in the first set belongs to the second set. Let x be an arbitrarily chosen element of $A \cup B$. Namely, let $x \in A \cup B$. We aim to prove that $x \in C \cup B$.

It's given that $x \in A \cup B$.

By definition of set union, either we have that $x \in A$ or $x \in B$. (Of course, it's possible that both are true, but we know that at least one must be true.)

If the latter is true, by the definition of union, it follows that $x \in C \cup B$.

Thus, we must only consider the case where $x \notin B$, since we've already proven the claim for all $x \in B$. In this case, since we know that $x \in A$ or $x \in B$ is true, it follows that $x \in A$ (Disjunctive Syllogism). Thus, we have that $x \in A$ and $x \notin B$. By definition of set difference, we know that $x \in A - B$. Since it's given that $A - B = C - B$, it follows that $x \in C - B$, by definition of set equality. Then, by definition of set difference, we have that $x \in C$ and $x \notin B$. Since $x \in C$, in this case, by definition of union, it follows that $x \in C \cup B$, as desired.

Thus, we've proven in all cases of the arbitrarily chosen element x , that $x \in C \cup B$, proving the assertion for all x .

Grading: Starting with arbitrary element - 1 pt

Stating there are two cases, x in A or x in B - 1 pt

Handling proof for x in B - 1 pt

Showing that only case left is x in A and not in B - 1 pt

Showing in this case that x is in $C - B$, so that it's in C - 2 pts

Concluding the proof that x is in C union B - 1 pt

3) (10 pts) Let R be the following relation defined over the set of integers:

$$R = \{ (a, b) \mid (a + b) \equiv 0 \pmod{10} \}$$

Is R reflexive, irreflexive, symmetric, anti-symmetric or transitive?

Please give your answer (yes/no) for each of the five parts and give a brief justification for each. Each answer is worth 1 pt and each justification is worth 1 pt. You can not earn the point for the justification if the answer is incorrect.

Solution

The relation is NOT reflexive, since $(3,3) \notin R$, because $3 + 3 \equiv 6 \pmod{10}$.

The relation is NOT irreflexive since $(5,5) \in R$, because $5 + 5 \equiv 0 \pmod{10}$.

The relation IS symmetric. Let (a, b) be an arbitrary element of the relation. Then we know that $(a + b) \equiv 0 \pmod{10}$, but it also follows from this that $(b + a) \equiv 0 \pmod{10}$, since $a + b = b + a$. The latter statement infers that (b, a) is in the relation, as desired.

The relation is NOT anti-symmetric because both $(3, 7)$ and $(7, 3)$ are part of it.

The relation is NOT transitive because although both $(3, 7)$ and $(7, 3)$ are elements of it $(3, 3)$ is not an element of it.

Grading: 1 pt for each answer. If answer is correct, then 1 pt for the reasoning.

4) (5 pts) Let $f(x) = x^3 + ax^2 + bx + c$. It is known that $f(-1) = 84$, $f(0) = 72$ and $f(1) = 50$, what are the values of a , b and c ?

Solution

$$f(0) = 0^3 + a(0)^2 + b(0) + c = c = 72, \text{ so } \underline{\mathbf{c = 72.}}$$

$$f(1) = 1^3 + a(1)^2 + b(1) + 72 = a + b + 73 = 50$$

$$f(-1) = (-1)^3 + a(-1)^2 + b(-1) + 72 = a - b + 71 = 84$$

Adding these two equations we get:

$$2a + 144 = 134$$

$$2a = -10$$

$$\underline{\mathbf{a = -5}}$$

Plugging into the first equation and solving for b we find:

$$-5 + b + 73 = 50$$

$$b + 68 = 50$$

$$\underline{\mathbf{b = -18}}$$

it follows that $f(x) = x^3 - 5x^2 - 18x + 72$. Incidentally, this factors nicely: $f(x) = (x-3)(x+4)(x-6)$.

Grading: 1 pt for figuring out c

2 pts for plugging in $x = 1$, $x = -1$

1 pt for value of a

1 pt for value of b