<u>COP 3502 Section 2 Exam #2 - Part 2 (Mathematical Induction) - 45 pts Solution</u>

Date: 10/22/2020

Start Time: 2:00 pm EST
End Time: 2:45 pm EST

1) (12 pts) Using induction on n, prove the following inequality for all positive integers n:

$$\sum_{i=1}^{2^n-1} log_2 i \geq (n-2)2^n + 2$$

<u>Solution</u>

Using induction on n, we will prove the given assertion.

Base case: n = 1, LHS = $\sum_{i=1}^{2^1-1} log_2 i = log_2 1 = 0$, RHS = $(1-2)2^1 + 2 = -2 + 2 = 0$
Thus, for n = 1 the formula holds and the base case is true.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that:

$$\sum_{i=1}^{2^k-1} log_2 i \geq (k-2)2^k + 2$$

Inductive step: Prove for n = k+1 that:

$$\sum_{i=1}^{2^{k+1}-1} log_2 i \geq ((k+1)-2)2^{k+1} + 2 = (k-1)2^{k+1} + 2$$

$$\sum_{i=1}^{2^{k+1}-1} log_2 i = \left[\sum_{i=1}^{2^k-1} log_2 i\right] + [\sum_{i=2^k}^{2^{k+1}-1} log_2 i]$$

$$\geq (k-2)2^k + 2 + [\sum_{i=2^k}^{2^{k+1}-1} log_2 i]$$, using inductive hypothesis

Note that the smallest term in the remaining sum is the first term, $log_2(2^k) = k$, since the log function is increasing. Also note that the number of terms in the sum is $2^{k+1}-1 - 2^k + 1 = 2^{k+1} - 2^k = 2^k$. Utilizing this information, we can extend the inequality as follows:

$$\geq (k-2)2^k + 2 + k(2^k)$$
$$= 2^k(k-2+k) + 2$$
$$= 2^k(2k-2) + 2$$

$$= (2^k)(2)(k-1) + 2$$
$$= (2^{k+1})(k-1) + 2$$
$$= (k-1)2^{k+1} + 2$$

This proves the inductive step. Thus, we can conclude for all positive integers n that

$$\sum_{i=1}^{2^n-1} log_2 i \geq (n-2)2^n + 2$$

Grading: Base Case - 1 pt
Inductive Hypothesis - 1 pt
Inductive Step - 2 pts
Sum Split - 2 pts
Use of IH - 2 pts
Bound last sum - 2 pts
End Algebra - 2 pts

2) (15 pts) Define the sequence t_n for positive integers n as follows:

$$t_1 = t_2 = t_3 = 1$$
$$t_n = t_{n-1} + t_{n-2} + t_{n-3}, for\ n > 3$$

Using strong induction on n with four base cases, prove the following for all positive integers n.

if $4 \mid n$, then $t_n \equiv 3\ mod\ 4$
if $4 \nmid n$, then $t_n \equiv 1\ mod\ 4$

Note: In your proof you will have four cases, but three of those four cases turn out to be equivalent, so it's enough to break your proof into two cases: where k+1 is divisible by 4 and where k+1 is NOT divisible by 4.

<u>Solution</u>
Use strong induction on n to prove the claim.

Base cases: n = 1, 2, 3, 4. The first four terms of the sequence are 1, 1, 1 and 3. As can be seen, the first three terms, when n is NOT divisible by 4 are equivalent to 1 mod 4, since all are equal to 1 and when n = 4, the corresponding term $t_4 \equiv 3$ (mod 4), as desired. So the given assertion is true for n = 1, 2, 3, 4, proving the first four base cases. **(Note: we only need 3 base cases, actually…)**

Inductive hypothesis: Assume for all positive integers n ≤ k, where k is an arbitrarily chosen integer 4 or greater, that

if $4 \mid n$, then $t_n \equiv 3\ mod\ 4$
if $4 \nmid n$, then $t_n \equiv 1\ mod\ 4$

Inductive step: Prove for n = k + 1 that

if $4 \mid (k+1)$, then $t_{k+1} \equiv 3 \; mod \; 4$
if $4 \nmid (k+1)$, then $t_{k+1} \equiv 1 \; mod \; 4$

Let's break our work into two cases, one case where $4 \mid (k+1)$ and another case where $4 \nmid (k+1)$.

<u>Case: k+1 is divisible by 4</u>

$$t_{k+1} = t_k + t_{k-1} + t_{k-2}$$

Since k+1 is divisible by 4, it follows that k-2, k-1 and k are not. (Specifically, we know that k ≡ 3 (mod 4), (k-1) ≡ 2 (mod 4) and (k-2) ≡ 1 (mod 4).) Due to our strong inductive hypothesis, we can guarantee that we know that each of the three terms on the RHS are equivalent to 1 (mod 4). Thus, using the inductive hypothesis, we have:

$\equiv 1 + 1 + 1 \; (mod \; 4)$, using the strong inductive hypothesis 3 times

$\equiv 3 \; (mod \; 4)$

This completes the proof of this case, since we were trying to show that $t_{k+1} \equiv 3 \; (mod \; 4)$

<u>Case: k+1 is NOT divisible by 4</u>

$$t_{k+1} = t_k + t_{k-1} + t_{k-2}$$

Since k+1 is NOT divisible by 4, it follows that exactly one of k-2, k-1 and k are divisible by 4. (To see this, note that k+1 ≡ 1 (mod 4) or k+1 ≡ 2 (mod 4) or k+1 ≡ 3 (mod 4). Subtracting appropriately for each equation yields: k ≡ 0 (mod 4) or k-1 ≡ 0 (mod 4) or k-2 ≡ 0 (mod 4). Thus, of the given terms, using the strong inductive hypothesis, exactly one of these terms (we don't know which) is equivalent to 3 (mod 4) and the other two terms are equivalent to 1 (mod 4).

$\equiv 3 + 1 + 1 \; (mod \; 4)$, using the strong inductive hypothesis 3 times

$\equiv 1 \; (mod \; 4)$

This completes the proof of this case, since we were trying to show that $t_{k+1} \equiv 1 \; (mod \; 4)$, when k+1 is not divisible by 4.

Grading: Base Cases: 2 pts
Inductive Hypothesis: 2 pts (give 1 if the swap variables)
Inductive Step: 2 pts (make sure they have this in k)
First Breakdown Step: 2 pts
Divisible case arguing 3 terms are 1: 3 pts
Non-divisible case arguing 1 term is 3 others are 1 and adding: 4 pts

3) (15 pts) Prove, using strong induction with four base cases, that for all positive integers n ≥ 24, that n can be expressed as the sum of perfect squares, not including 1. (Note: perfect squares are numbers that can be represented as an integer squared. The first few perfect squares except for 1 are 4, 9, 16, and 25.) **Also, prove that 24 is the smallest integer for which such a claim can be made.**

Solution

Use strong induction on n with four base cases to prove the claim:

Base cases: n = 24, 25, 26, and 27.

24 = 16 + 4 + 4
25 = 16 + 9
26 = 9 + 9 + 4 + 4
27 = 9 + 9 + 9

Inductive hypothesis: Assume for all positive integers n, 24 ≤ n ≤ k, where k is an arbitrarily chosen positive integer 27 or greater, that all integers n in this range can be expressed as a sum of squares (except for 1).

Inductive step: Prove for n = k+1 that it can be expressed as the sum of perfect squares.

k+1 = (k-3) + 4
= [valid sum of squares] + 4, using the strong inductive hypothesis.
= valid sum of squares.

Since we know that k ≥ 27, it follows that k-3 ≥ 24, so that we can apply the strong inductive hypothesis to k-3. Since 4 is a perfect square, we've found a way to express k+1 as a valid sum of squares, as desired. Thus, the inductive step is proven and we can conclude for all integers 24 or greater, they can be expressed as the sum of perfect squares without using 1.

To prove that 24 is the smallest such integer for which we can make the claim, note that under 24, the only perfect squares we are allowed to use are 4, 9 and 16. But note that two of these are equivalent to 0 (mod 4) and 1 is equivalent to 1 (mod 4). Now, note that 23 ≡ 3 (mod 4). Thus, in order for any set of values from {4, 9, 16} to add up to 23, at a minimum, we would have to include 9 three times, since neither 4 nor 16 contribute to the sum mod 4. But, 9 + 9 + 9 = 27, so it follows that there is no solution to adding values from the set {4, 9, 16} to equal exactly 23. (Also, a quick brute force search can prove this assertion. You could argue that if we had 16, then we need to add to 7, which is impossible since 4 is the only number left. This means there is no 16 in the solution. Then try 9. If we try 2 9s, we are left with trying to get 5, which isn't possible. If we try 1 9, then we have to get exactly 14 with 4s only, which also isn't possible. Finally, it's impossible to add to 23 using 4s only. This means that no solution is possible for 23.)

Grading: base cases: 8 pts, 2 pts for each break down
Inductive hypothesis: 2 pts
Inductive step: 1 pt
Splitting off 4 from k+1: 3 pts
Concluding the proof: 1 pt

4) (3 pts) The book, "The Invisible Gorilla", talks about a famous experiment where many subjects did not see a hairy primate. It turns out that the primate in question was a human in a suit. What type of primate was the human actor in a suit attempting to mimic? (Note: The book talks about a lot of other experiments as well, and isn't necessarily focused on the one experiment mentioned in the question.)

Gorilla (https://www.youtube.com/watch?v=IGQmdoK_ZfY)

Grading: Give to All