COP 3502 Section 2 Exam #2 - Part 1 (Number Theory, Recitation Topics) - 30 pts Solution

Date: 10/22/2020

Start Time: 1:30 pm EST
End Time: 2:00 pm EST

1) (10 pts) Use the Extended Euclidean Algorithm to determine 43^{-1} mod 95. Points will be awarded for the process only and not the answer. To receive credit, you must use the process shown in class.

Solution

Run the Euclidean Algorithm:

95 = 2 x 43 + 9
43 = 4 x 9 + 7
9 = 1 x 7 + 2
7 = 3 x 2 + 1, so gcd(95, 43) = 1

Now, run the Extended Euclidean Algorithm:

7 - 3 x 2 = 1
7 - 3(9 - 7) = 1
7 - 3 x 9 + 3 x 7 = 1
4 x 7 - 3 x 9 = 1
4(43 - 4 x 9) - 3 x 9 = 1
4 x 43 - 16 x 9 - 3 x 9 = 1
4 x 43 - 19 x 9 = 1
4 x 43 - 19(95 - 2 x 43) = 1
4 x 43 - 19 x 95 + 38 x 43 = 1
42 x 43 - 19 x 95 = 1

Now, take this equation mod 95 to yield

42 x 43 - 19 x 95 ≡ 1 (mod 95)
42 x 43 - 19 x 0 ≡ 1 (mod 95)
42 x 43 - 0 ≡ 1 (mod 95)
42 x 43 ≡ 1 (mod 95)

It follows that $43^{-1} \equiv$ **42 (mod 95).**

Grading: 3 pts Euclidean, 5 pts Extended, 1 pt taking mod 95, 1 pt extracting answer.

2) (10 pts) There are 10 quizzes in a class which comprise the whole grade for the class. The i^{th} quiz is out of 10i points and each point in the class has equal value. (Note: the quizzes do NOT have equal value…) After the first 8 quizzes, Tamara has an 80% in the class. Note that all scores on quizzes are an integer number of points. What is the highest integer percentage average she can receive for the 10 quizzes? If she achieved this average exactly, but didn't ace (earn all of the points) either the 9^{th} or 10^{th} quiz, what is the sum of the digits of her scores on the last two quizzes? Also, given this information, are both quiz scores uniquely determined, or are there multiple pairs of scores for quiz 9 and 10 that satisfy all the given constraints. Prove your answer.

Solution

The number of points available on the first 8 quizzes is $\sum_{i=1}^{8} 10i = 10\sum_{i=1}^{8} i = 10\left(\frac{8\times 9}{2}\right) = 360$.
Tamara has earned .8 x 360 = 288 of these points.

Note that the total number of points of all quizzes is $\sum_{i=1}^{10} 10i = 10\sum_{i=1}^{10} i = 10\left(\frac{10\times 11}{2}\right) = 550$.

Let x be the total number of points she earns after 10 quizzes and let y be her integer percentage in the class. Note that both x and y must be integers based on the problem specification. This gives us the following equation:

$$\frac{x}{550} = \frac{y}{100}$$

$$100x = 550y$$
$$2x = 11y$$

The quantity on the right is divisible by 11. Thus, 11 | 2x. Since gcd(11, 2) = 1, it follows that 11 | x.

Also, we know that Tamara earned 288 points so far. Let z equal the number of points she earns out of 190 on the last two quizzes. We then have the equation:

288 + z = x, where 11 | x, and z ≤ 190.

Take this equation mod 11 and we get:

288 + z ≡ x (mod 11)
2 + z ≡ 0 (mod 11)
z ≡ -2 ≡ 9 (mod 11)

Recalling that 187 ≡ 0 (mod 11), it follows that the largest integer equivalent to 9 (mod 11) under 190 is 185. Thus, she scored 185 points out of 190 points on the last two quizzes, for a total average of (288+185)/5.5 = **86%.** (She scored 473 total points out of 550.) If she didn't ace either quiz, then her possible scores were (86, 99), (87, 98), (88, 97) or (89, 96), removing the two possibilities where she scored a 90 on quiz 9 or a 100 on quiz 10. In all four of these cases, the sum of digits is 8 + 6 + 9 + 9 = **32.** Both quiz scores aren't uniquely determined. All four ordered pairs given above are consistent with the given information.

Grading: 2 pts figuring out 360 pts in first 8 quizzes, 1 pt for figuring out she has 288 now. 5 pts for figuring out she can't exceed 86% and have an integer percentage. 1 pt for getting the digit sum. 1 pt for showing two different possibilities that achieve the same digit sum.

Alternate Solution Idea for #2

Use the work from the first solution to determine that there are 360 points for the first eight quizzes and 550 points for all 10 quizzes.

Tamara loses 20% of 360 points which equals 72 points. This means her maximum grade for all the quizzes is $\frac{550-72}{550} = \frac{478}{550} < \frac{478.5}{550} = 87\%$. Thus, it would be impossible for her to earn an 87% average in the class. Let's see if she can earn an 86% average. .86 x 550 = 473. Since this is an integer number of points, she can earn this average, by getting 473 - 288 = 185 points on the last two quizzes. Then, continue with the rest of the solution in the first solution idea.

Grading: Same as the first idea - this solution shows a different way to reason out that 86% is the highest integer percentage she could earn.

3) (10 pts) What is the sum of divisors of the integer 34.3 million? Please give your answer in prime factorized form. (Note: 5 points for any reasonable expression that is correct, 5 points to take that expression and represent it in prime factorized form.) **Note: In order to earn full credit, you must show each step as if you didn't use a calculator. (Namely, you may use the calculator to check basic arithmetic, but each step must be something I could imagine a reasonable student doing without a calculator. In short, I am specifically testing skills I went over that allow you to solve this problem completely without a calculator.)**

Solution

$34{,}300{,}000 = 343 \times 10^5 = 7^3 \times 2^5 \times 5^5$

Using the formula given for the sum of divisors for this number in class, we have:

$$\frac{(7^4-1)}{(7-1)} \times \frac{(2^6-1)}{(2-1)} \times \frac{(5^6-1)}{(5-1)} =$$

$$\frac{(7^2-1)(7^2+1)}{2 \times 3} \times \frac{(2^3-1)(2^3+1)}{1} \times \frac{(5^3-1)(5^3+1)}{2 \times 2} =$$

$$\frac{(48)(50)}{6} \times \frac{(7)(3^2)}{1} \times \frac{(124)(126)}{2 \times 2} =$$

$$8 \times 50 \times 7 \times 3^2 \times 31 \times 126 =$$
$$2^3 \times 2 \times 5^2 \times 7 \times 3^2 \times 31 \times 2 \times 63 =$$
$$2^3 \times 2 \times 5^2 \times 7 \times 3^2 \times 31 \times 2 \times 7 \times 3^2 =$$
$$\mathbf{2^5 \times 3^4 \times 5^2 \times 7^2 \times 31}$$

Grading: 5 pts for writing down correctly what is on the first line of the solution, 5 pts to reduce to final prime factorization. Give partial on the last part as follows: (2 pts to use square difference or cube difference formulas, 3 pts for all the arithmetic to simplify. Don't give these 5 pts if it looks like they plugged into a calculator and skipped factoring.)