Fall 2019 COT 3100 Homework 9
Please Consult WebCourses for the due date/time

Note: Please *justify* your answers and why you use each formula.

1) An inversion in a permutation of the integers 1 to n is a pair of numbers (not necessarily adjacent) such that the larger number is listed first. For example, in the permutation 4, 2, 3, 1, the inverted pairs are (4, 2), (4, 3), (4, 1) (2, 1) and (3, 1). By listing out all 24 permutations and counting the number of inversions in each (if you are lazy you can write a program to do this and attach the code as a separate file), calculate the expected number of inversions in a random permutation of 1, 2, 3 and 4. Then, using this result, posit a guess for the general result, in terms of n for permutations of 1, 2, 3, …, n. Try to prove this guess via a route that uses less calculation, but looks at an arbitrary pair of indexes into the permutation, say i and j with $i < j$ and counts how many permutations for which this pair is "in order" and that this pair is in inverted.

2) A Bubble Sort is a common algorithm taught to students that sorts a list of numbers (https://en.wikipedia.org/wiki/Bubble_sort). Given a random permutation of the integers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10, what is the probability that the permutation will be sorted **just one pass** of Bubble Sort? (For example, the permutation 3, 1, 2, 7, 4, 5, 6, 10, 8, 9 would get sorted in a single pass. 3 would swap with 1 and 2. Then 7 would sway with 4, 5 and 6, followed by 10 being swapped by 8 and 9. But, the permutation 3, 1, 2, 7, 4, 5, 8, 6, 10, 9 would not get sorted by one pass of the algorithm. After one pass, the array would be 1, 2, 3, 4, 5, 7, 6, 8, 9, 10.)

3) A divisor of 15^{50} is randomly selected. What is the probability that the selected divisor is a multiple of $3^{30}5^{20}$?

4) Let R_1 and R_2 be relations on a set A = {1, 2, 3, 4}.
In particular, let R_1 = {(1, 3), (1, 4), (2, 1), (2, 2), (2, 4), (3, 4), (4, 3) (4, 4)} and R_2 = {(1, 2), (1, 3), (2, 4), (3, 4), (4, 1)}

Determine the following:

a) Whether or not R_1 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
b) Whether or not R_2 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
c) The relation $R_1 \circ R_2$.
d) The relation $R_2 \circ R_1$.
e) $R_1 \cup R_2$
f) $R_1 \cap R_2$
g) The reflexive, symmetric and transitive closures of both R_1 and R_2.

5) Let R be a relation over the positive integers defined as follows:

R = {(a,b) | gcd(a,b) = min(a, b)}

In laymen's terms, describe how to determine whether or not two positive integers are related via R.

Determine whether or not R satisfies the following properties. Give a brief justification for each of your answers.

(i) reflexive
(ii) irreflexive
(iii) symmetric
(iv) anti-symmetric
(v) transitive

6) How many anti-symmetric relations on the set A = {1, 2, 3, 4, 5, 6, 7, 8} contain the ordered pairs (2, 3), (5, 2), (3, 3), (4, 4), (6, 6), (7, 8), (8, 4) and (8, 8)?

7) Let the relation R, over the positive integers, be defined as follows:

R = { (a, b) | b = an, for some positive integer n }

Prove that R is a partial ordering relation.

8) Give a summary of the mathematical contributions of Srinivasa Ramanujan. Please aim for a length of roughly 200 - 400 words. **Your summary must be typed.** Please state the sources you used in writing your summary.