Fall 2019 COT3100 Homework - 9

Solutions

Problem 1. An inversion in a permutation of the integers 1 to n is a pair of numbers (not necessarily adjacent) such that the larger number is listed first. For example, in the permutation 4, 2, 3, 1, the inverted pairs are (4, 2), (4, 3), (4, 1) (2, 1) and (3, 1). By listing out all 24 permutations and counting the number of inversions in each (if you are lazy you can write a program to do this and attach the code as a separate file), calculate the expected number of inversions in a random permutation of 1, 2, 3 and 4. Then, using this result, posit a guess for the general result, in terms of n for permutations of 1, 2, 3, â, n. Try to prove this guess via a route that uses less calculation, but looks at an arbitrary pair of indexes into the permutation, say i and j with $i <$j and counts how many permutations for which this pair is "in order" and that this pair is in inverted.

Proof. Let i, j be indices in the array 1, 2, 3, ..., n such that $1 \leq i < j \leq n$. Let I_{ij} represent whether or not an inversion has occurred. That is to say, $I_{ij} = 1$ iff $arr[i] > arr[j]$, otherwise $I_{ij} = 0$. Therefore, the total number of inversions is $\sum I_{ij}$. Because we are choosing permutations randomly, the probability that $arr[i] > arr[j]$ is equal to the probability that $arr[i] < arr[j]$. Therefore, $E[I_{ij}] = \frac{1}{2}$.

The number of ways to uniquely pair of a list of integers 1 to n is simply the sum of integers 1 to (n - 1). The first number can pair up with every other number (which is n - 1 options). The second can pair up with every number except itself and the first, which is n - 2 options, and so on. We know that the sum of integers 1 to n - 1 is $\frac{n(n-1)}{1}$. We can therefore calculate our answer.

$$E[\sum_{pairs} I_{ij}] = \sum_{pairs} E[I_{ij}] = \sum_{pairs} \frac{1}{2} = \frac{n(n-1)}{2} * \frac{1}{2} = \frac{n(n-1)}{4}$$

□

Problem 2. A Bubble Sort is a common algorithm taught to students that sorts a list of numbers. Given a random permutation of the integers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10, what is the probability that the permutation will be sorted just one pass of Bubble Sort? (For example, the permutation 3, 1, 2, 7, 4, 5, 6, 10, 8, 9 would get sorted in a single pass. 3 would swap with 1 and 2. Then 7 would sway with 4, 5 and 6, followed by 10 being swapped by 8 and 9. But, the permutation 3, 1, 2, 7, 4, 5, 8, 6, 10, 9 would not get sorted by one pass of the algorithm. After one pass, the array would be 1, 2, 3, 4, 5, 7, 6, 8, 9, 10.)

Proof. Because bubble sort always terminates when the array is sorted, we can define an array sortable with one pass as any array that causes bubble sort to terminate in one pass. The number of arrays that do this can therefore be found by the number of different ways that bubble sort can take place in one pass. For any array of size n, there are n-1 attempted swaps in a pass of bubble sort. During that pass, each of these swaps either occurs or does not occur, and therefore there are 2^{n-1} possible combinations of swaps. The total number of arrays is simply a permutation, so n!. Therefore, the probability that a random array will be sortable in one pass of bubble sort is $\frac{2^{n-1}}{n!}$. In the case of an array of size 10, this is $\frac{2^9}{10!} = \frac{2}{14175}$. □

Problem 3. A divisor of 15^{50} is randomly selected. What is the probability that the selected divisor is a multiple of $3^{30}5^{20}$

Proof. Any divisor of 15^{50} is made up of some number of 3s (less than or equal to 50) and 5s (less than or equal to 50). There are 51 x 51 such divisors. Therefore, for any divisor of this number, we can only obtain multiples of $3^{30}5^{20}$ by multiplying it by some number of 3s and 5s, with atleast 30 3s and 20 5s. So, we can find the probability that a randomly selected divisor of 15^{50} is a multiple of this number by simply finding the probability that the number of 3s and number of 5s in said divisor is greater than or equal to 30 for 3, and 20 for 5. Therefore, the answer is $\frac{21}{51} * \frac{31}{51} = \frac{217}{867}$ □

Problem 4. Let R1 and R2 be relations on a set A = 1, 2, 3, 4. In particular, let R1 = (1, 3), (1, 4), (2, 1), (2, 2), (2, 4), (3, 4), (4, 3) (4, 4) and R2 = (1, 2), (1, 3), (2, 4), (3, 4), (4, 1)

Determine the following: a) Whether or not R1 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
b) Whether or not R2 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
c) The relation $R1 \circ R2$.
d) The relation $R2 \circ R1$.
e) $R_2 \bigcup R_2$
f) $R_1 \cap R_2$
g) The reflexive, symmetric and transitive closures of both R1 and R2.

Proof. a) R1 does not contain (1,1), so it is not reflexive. It contains (2, 2) so it is not irreflexive. It contains (1, 3), but not (3, 1), so is not symmetric. It contains (4, 3) and (3, 4), so it is not anti-symmetric. It contains (2, 1) and (1, 3), but does not contain (2, 3), so it is not transitive.
b) R2 does not contain a single pair in the form (a, a), so it is irreflexive, and therefore is cannot be reflexive. For any pair (a, b) in R2, (b, a) is not in R2, so it is anti-symmetric and therefore not symmetric. It contains (1, 2) and (2, 4) but not (1, 4), so it is not transitive.
c) {(1, 1),(1, 2),(1, 4),(2, 3), (2, 4), (3, 3), (3, 4), (4, 3), (4, 4)}
d) (1, 4), (1. 1), (2, 2), (2, 3), (2, 4), (2, 1), (3, 1), (4, 4), (4, 1)
e) {(1, 2), (1, 3), (1, 4), (2, 1),(2, 2), (2, 4), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)}
f) {(1, 3), (2, 4), (3, 4)}
g)
Reflexive R1: {(1, 1), (1, 3), (1, 4), (2, 1), (2, 2), (2, 4), (3, 3), (3, 4), (4, 3) (4, 4)}
Reflective R2: {(1, 1), (1, 2), (1, 3), (2, 2), (2, 4), (3, 3), (3, 4), (4, 1), (4, 4)}
Symmetric R1: {(1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 4), (3, 1), (3, 4), (4, 1), (4, 2), (4, 3) (4, 4)}
Symmetric R2: {(1, 2), (1, 3), (2, 1), (2, 4), (3, 1), (3, 4), (4, 1), (4, 2), (4, 3)}
Transitive R1: {(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 3) (4, 4)}
Transitive R2: {(1, 2), (1, 3), (1, 4), (2, 1), (2, 4), (3, 1), (3, 4), (4, 1), (4, 2), (4, 3)}

□

Problem 5. Let R be a relation over the positive integers as follows:
$R = \{(a, b)|gcd(a, b) = min(a, b)\}$

In laymen's terms, describe how to determine whether or not two positive integers are related via R.
Determine either or not R satisfies the following properties:

Proof. .
The relation R describes two numbers such that one is a multiple of the other.

i. Reflexive:
R is reflexive as all integers are multiples of themselves.

ii. Irreflexive:
R is NOT irreflexive because (2, 2) is in R sicne gcd(2, 2) = 2.

iii. Symmetric:
R is symmetric because gcd(a,b) is the same as gcd(b, a), as is min(a, b). Because of how R is defined, the order does not matter. Therefore, if (a, b) is in the relation, so is (b, a).

iv. Anti-symmetric:
R is NOT anti-symmetric because (2, 4) is in R and (4, 2) is in R but 2 and 4 aren't equal .

v. Transitive:
Consider (12, 3) and (3, 9). The gcd of 12 and 3 is 3, as is their minimum. The gcd of 3 and 9 is 3, as is their minimum. Therefore, both pairs are in R. However, the gcd of (12, 9) is 3, but their minimum is 9, so (12, 9) is not in R. Therefore, R is not transitive.

□

Problem 6. How many anti-symmetric relations on the set A = 1, 2, 3, 4, 5, 6, 7, 8 contain the ordered pairs (2, 3), (5, 2), (3, 3), (4, 4), (6, 6), (7, 8), (8, 4), and (8, 8)?

First, consider all pairs in the form (a, b) where a = b. Since there are 8 numbers, so there are 8 of these pairs. Four of them are required to be in the relation (3, 4, 6 and 8), therefore there are 4 remaining pairs. Each of these pairs can either be or not be in the relation, so there are 2^4 possibilities considering these types of pairs.

Now consider pairs where $a \neq b$, but do not consider order (that is to say, from the 8 numbers, choose 2). There are 28 of these pairs. 4 of these pairs are required to be in the relation, so we reduce this down to 24. For each of these 24 pairs, there are 3 options. For pair (a, b), we can include (a, b), (b, a), or not include it at all. We cannot include both since the relation is anti-symmetric. Therefore, for the pairs, there are 3^{24} possibilities.
Therefore, our final answer is $2^4 3^{24} = 4518872583696$

Problem 7. Let the relation R, over the positive integers, be defined as follows:
R = (a,b)—b = an, for some positive integer n
Prove that R is a partial ordering relation

Proof. .
a Reflexivity:
Let a be an arbitrary positive integer. a = a(1), therefore (a, a) is in R. Since a is arbitrary, R is reflexive.

b Anti-Symmetry:
Let a and b be arbitrary positive integers, and let (a, b) be in R. Therefore, b = an, where n is some positive integer. Therefore, $a = \frac{1}{n}a$. The only way that $\frac{1}{n}$ can be an integer is if n = 1. So, the only way for both (a, b) and (b, a) to be in R is for a = b. Therefore, R is anti-symmetric.

c Transitivity:
Let a, b, and c be arbitrary positive integers. Let (a, b) and (b, c) be in R. Therefore, there exists an n and an m such that b = an and c = bm. Therefore, c = (an)m. By the association, c = a(nm). Since the integers are closed under multiplication, nm is an integer. Therefore, (a, c) is in R.

□

Problem 8. Give a summary of the mathematical contributions of Srinivasa Ramanujan. Please aim for a length of roughly 200 - 400 words. Your summary must be typed. Please state the sources you used in writing your summary.

Srinivasa Ramanujan, (born December 22, 1887, Erode, Indiaâ”died April 26, 1920, Kumbakonam), was a Indian mathematician whose contributions to the theory of numbers include pioneering discoveries of the properties of the partition function.

Srinivasa Ramanujan is to mathematics what many consider Mozart to be for music. With minimal formal training in mathematics, Ramanujan derived some of the most complicated known mathematics and even new theorems, completely on his own.

Born in a small village outside of Madras, India in 1887, Ramanujan very quickly showed his affinity for mathematics. By age 11, soon after he had been introduced to formal mathematics in school, he was able to exhaust all of the knowledge of two college boarders who lived at his house. At the age of 16, he obtained a copy of a pure mathematics book and devoured its contents. Using this book as a starting point, he calculated the Euler-Masheroni constant to 15 decimal places. Soon after, Ramanujan attempted to attend college, but due to the fact that all he cared about was mathematics, he earned poor grades in other subjects. After completing school, he spent most of free time doing mathematics, but ultimately had to get a job. He landed a job as a clerk in a revenue department. Upon seeing Ramanujan's notebooks, his boss realized that he didn't deserve such a boring job.

Ramanujan's boss showed the notebooks to other mathematicians in the area. Eventually, these notebooks circulated among a few British mathematicians, several of whom felt that though Ramanujan may have some talent, he lacked formal training and rigor in his proofs. Finally, however, two famous British mathematicians, G. H. Hardy and J. E. Littlewood, felt that Ramanujan's genius deserved to be discovered by others. They offered to have Ramanujan come to England to do mathematical research with them. In 1914, at the age of 27, Ramanujan took a ship from India to England to join Hardy and Littlewood. In working with Hardy and Littlewood, they improved the rigor of the proofs of Ramanujan's results, as often times, Ramanujan relied on his instincts. In 1918, he was elected a Fellow of the Royal Society, one of the youngest mathematicians to do so and only the second Indian ever, at the time.

Throughout his life, Ramanujan experienced health problems. Though many thought he died of tuberculosis, some now believe that he actually died from amoebiasis, a treatable disease that was widespread in Madras at the turn of the 20th century. Ramanujan passed away on April 26, 1920. In six short years of work with Hardy and Littlewood, he produced a great quantity of results. Some of the well-known results he proved dealt with the problem of partitions, the number of ways to divide a positive integer into a set of positive integers. (For example, 4 can be partitioned as follows: 4, 2+2, 1+3, 1+1+2, and 1+1+1+1.), the Bernoulli numbers and coming up with series to generate many digits of pi quickly.

One of the most remarkable qualities of Ramanujan was his ability to discover new mathematics while being ignorant of modern mathematical developments. . He worked out the Riemann series, the elliptic integrals, hypergeometric series, the functional equations of the zeta function, and his own theory of divergent series. On the other hand, he knew nothing of doubly periodic functions, the classical theory of quadratic forms, or Cauchyâ™s theorem, and he had only the most nebulous idea of what constitutes a mathematical proof. Though brilliant, many of his theorems on the theory of prime numbers were wrong.

He was described as enthusiastic and eager, though diffident. He made jokes, sometimes at his own expense. He could talk about politics and philosophy as well as mathematics. He was never particularly introspective. In official settings he was polite and deferential and tried to follow local customs. His native language was Tamil, and earlier in his life he had failed English exams, but by the time he arrived in England, his English was excellent. He liked to hang out with other Indian students, sometimes going to musical events, or boating on the river. Physically, he was described as short and stout â" with his main notable feature being the brightness of his eyes. He worked hard, chasing one mathematical problem after another.

Unfortunately, he died young at the age of 32 in his home in India. It is now known that we was developing new mathematics to the day he died, many were found much time after his death.

https://www.britannica.com/biography/Srinivasa-Ramanujan
https://www.wired.com/2016/04/who-was-ramanujan/
https://en.wikipedia.org/wiki/Srinivasa$_{R}amanujan$