COT 3100 Fall 2019 Homework #6 Solutions

1) Let H_n denote the n^{th} Harmonic number. (Recall that $H_n = \sum_{i=1}^{n} \frac{1}{i}$.)

Use mathematical induction on n to show that $H_{2^n} \geq 1 + \frac{n}{2}$, for all non-negative integers n.

Solution

Base case n = 0. LHS = $H_{2^0} = H_1 = 1$, RHS = $1 + \frac{0}{2} = 1$, thus the base case holds.

Inductive hypothesis: Assume for an arbitrarily chosen non-negative integer n = k that

$$H_{2^k} \geq 1 + \frac{k}{2}$$

Inductive step: Prove for n = k+1 that

$$H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$$

$$H_{2^{k+1}} = \sum_{i=1}^{2^k} \frac{1}{i} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i}$$

$$= H_{2^k} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i}$$

$$\geq 1 + \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i}, \text{ using the IH}$$

Noting that each term in the second sum is greater than or equal to $\frac{1}{2^{k+1}}$, we get:

$$\geq 1 + \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{2^{k+1}}$$

$$= 1 + \frac{k}{2} + 2^k \times \frac{1}{2^{k+1}}$$

$$= 1 + \frac{k}{2} + \frac{1}{2}$$
$$= 1 + \frac{k+1}{2}, \text{ proving the inductive step}$$

Since the inductive step is proven, we can conclude that $H_{2^n} \geq 1 + \frac{n}{2}$, for all non-negative integers n.

2) Let $f(n) = \frac{n}{n+2}$. Define $f^k(n)$ to be the function f composed with itself k times. More formally, $f^0(n) = n$ and $f^k(n) = f(f^{k-1}(n))$, for all positive integers k. Using induction on k, prove that for all positive integers k, $f^k(n) = \frac{n}{(2^k-1)n+2^k}$. (Hint: The algebra can be messy if you don't multiply both your numerator and denominator by $(2^k - 1)n + 2^k$. So, in full, after you do a particular step, you would take your fraction and multiply it by $\frac{(2^k-1)n+2^k}{(2^k-1)n+2^k}$. Please feel free to ignore the hint, but I do think it reduces the amount of algebra drastically.) (Note: This is an exam question from a previous semester, so a good question for practice!)

Solution

Base case: n = 1, LHS = $f^1(n) = \frac{n}{n+2}$, given. RHS = $\frac{n}{(2^1-1)n+2^1} = \frac{n}{n+2}$, proving the base case.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer k = m that:

$$f^m(n) = \frac{n}{(2^m-1)n+2^m}.$$

Inductive step: Prove for k = m+1 that $f^{m+1}(n) = \frac{n}{(2^{m+1}-1)n+2^{m+1}}$.

$$\begin{aligned}
f^{m+1}(n) &= f(f^m(n)) \\
&= f\left(\frac{n}{(2^m-1)n+2^m}\right) \\
&= \frac{\frac{n}{(2^m-1)n+2^m}}{\frac{n}{(2^m-1)n+2^m}+2} \\
&= \frac{n}{n+2((2^m-1)n+2^m)} \\
&= \frac{n}{n+2^{m+1}n-2n+2^{m+1}} \\
&= \frac{n}{-n+2^{m+1}n+2^{m+1}} \\
&= \frac{n}{(2^{m+1}-1)n+2^{m+1}} \text{ , proving the inductive step}
\end{aligned}$$

Since the inductive step has been proven, we can conclude that $f^k(n) = \frac{n}{(2^k-1)n+2^k}$ for all positive integers k.

3) Consider a rectangular prism with a total surface area of 94 in^2. If the sum of all of its edges is 48 in, what is the sum of the lengths of all of its interior diagonals, in inches? Based on the given information, can we determine the exact dimensions of the prism? Why or why not?

Solution

total surface area $= 2xy + 2xz + 2yz = 94$

sum of edges $= 4x + 4y + 4z = 4(x + y + z) = 48$

$$x + y + z = 12$$

One interior diagonal: $\sqrt{\left(\sqrt{x^2 + y^2}\right)^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$

$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz = 144$

$$x^2 + y^2 + z^2 + 94 = 144$$

$$x^2 + y^2 + z^2 = 50$$

One interior diagonal $= \sqrt{x^2 + y^2 + z^2} = \sqrt{50}$

There are 4 interior diagonals and they are of equal length.

sum of interior diagonals $= 4\sqrt{x^2 + y^2 + z^2} = 4\sqrt{50} = \mathbf{20\sqrt{2}}\ \boldsymbol{inches}$

No, you cannot find the dimensions, since you have 3 unknown variables and are only given 2 equations. (Some tinkering can produce multiple solutions. (3, 4, 5) is one solution and $(3.5, \frac{17}{4} - \frac{\sqrt{13}}{4}, \frac{17}{4} + \frac{\sqrt{13}}{4})$ is another, which proves that a solution can not be uniquely determined.)

4) Let P be a cubic polynomial such that P(0) = k, P(1) = 2k, and P(-1) = 3k. What is P(2) + P(-2)? Based on the given information, is the polynomial P uniquely determined? Why or why not?

Solution

A cubic polynomial will be in the form: $P(x) = ax^3 + bx^2 + cx + d$

$P(0) = a(0)^3 + b(0)^2 + c(0) + d = d = k$

$P(1) = a + b + c + d = 2k \qquad a + b + c + k = 2k \qquad a + b + c = k$

$P(-1) = -a + b - c + d = 3k \quad -a + b - c + k = 3k \quad -a + b - c = 2k$

$$P(1) + P(-1) = a + b + c - a + b - c = k + 2k \qquad 2b = 3k \qquad b = \frac{3}{2}k$$

$$P(2) \;=\; a(2)^3 + (2)^2 b + c(2) + d \quad = \quad 8a + 4b + 2c + d$$

$$P(-2) = a(-2)^3 + (-2)^2 b + c(-2) + d = \; -8a + 4b - 2c + d$$

$$P(2) + P(-2) = 8a + 4b + 2c + d - 8a + 4b - 2c + d = 8b + 2d = 8\left(\frac{3}{2}k\right) + 2(k)$$

$$= 12k + 2k = \mathbf{14k}$$

No, P cannot be uniquely determined because you cannot find specific values for a or b with the given information.

5) How many permutations of MISSISSIPPI satisfy the following constraints? Consider each case independent of the others.

a) Have all of its consonants preceding all of its vowels.
b) Have no two consecutive vowels.
c) Start and end with a vowel.
d) Start and end with a consonant.

Solution

a)

An example permutation would be MSSSSPPIIII, the side with the vowels will stay the same since I is our only vowel, so we only have to find all the permutations of the consonants, MSSSSPP.

$$\mathbf{\frac{7!}{1! * 4! * 2!}}$$

b)

The vowels have to be separated by consonants, so the possible positions for vowels would be of the form: _M_S_S_S_S_P_P_, where MSSSSPP is only one of the many possible permutations of consonants.

We have 8 spots to put vowels and only have 4 vowels, so the number of combinations of positions is $\binom{8}{4}$.

We also should consider the permutations of the consonants as well, which is what we got in (a).

The vowels only have 1 permutation, since they are all I.

Each permutation of the consonants will have $\binom{8}{4}$ different combinations, so we multiply.

$$\binom{8}{4} * \frac{7!}{1! * 4! * 2!}$$

c)

This is the same as removing two I's , since you know that one I will always be in the front and one I will always be at the end. We are essentially permuting MSSSSPPII

$$\frac{9!}{1! * 4! * 2! * 2!}$$

d)

Same idea in (c) except we have multiple unique consonants, so we have to consider it case by case.

There is a separate case for every two consonants we choose to put at the beginning and end.

We have to add up all of the number of permutations that result from each case.

The different possible cases would be M…S, S…M, M…P, P…M, P…P, and S…S.

Perm of M…S = Perm of S…M $= \frac{9!}{4!*3!*2!}$

Perm of M…P = Perm of P…M $= \frac{9!}{4!*4!}$

Perm of P…S = Perm of S…P $= \frac{9!}{4!*3!}$

Perm of P…P $= \frac{9!}{4!*4!}$

Perm of S…S $= \frac{9!}{4!*2!*2!}$

Total # of permutations $= 2 * \frac{9!}{4!*3!*2!} + 3 * \frac{9!}{4!*4!} + 2 * \frac{9!}{4!*3!} + \frac{9!}{4!*2!*2!}$

$$= 2 * \frac{9!}{24*12} + 3 * \frac{9!}{24*24} + 2 * \frac{9!}{24*6} + \frac{9!}{24*4}$$

$$= \frac{9!}{24*6} + \frac{9!}{24*8} + \frac{9!}{24*3} + \frac{9!}{24*4}$$

$$= \frac{9!}{24}\left(\frac{1}{6} + \frac{1}{8} + \frac{1}{3} + \frac{1}{4}\right) = \frac{9!}{24}\left(\frac{4+3+8+6}{24}\right) = 9(2)(7)(5)(21) = 13,230$$

6) How many ways can we place 8 rooks on a standard 8 by 8 chessboard such that no two rooks can attack one another? (If necessary, please look up the rules of chess to help you solve this problem.)

Solution

A rook can attack another one if it's in the same row or column. This means each rook must be the only one its column and row. For the first row you have the full 8 choices of columns to put it. For the second row you only have 7 choices, since you know that one column already has a rook in it. Then for the third row you have six, since two columns are occupied. This pattern continues until the last rook only has one choice because all the other columns have rooks in them.

You should multiply all these choices: 8*7*6*5*4*3*2*1 = **8! Ways**

7) Give a summary of the life and mathematical contributions of Emmy Noether. Please aim for a length of roughly 200 - 400 words. ***Your summary must be typed.*** Please state the sources you used in writing your summary.

Amalie Emmy Noether was born in 1882 in Erlangen, Germany to a Jewish family. Her father, Max Noether, was a mathematician, leading Emmy to study mathematics at the University of Erlangen, where her father was a lecturer. After completing her dissertation, Noether went to work at the Mathematical Institute of Erlangen for seven years, where she made no income. In 1915, she joined the Universiry of Gottingen, with great resisstence from the faculty due to her being a woman. Mathemetician B. L. van der Waerden would become aqcuatences with Noether and use her as a source for a lot of the landmark material that he would go on to publish, such as his 1931 textbook, *Moderne Algebra*. Noether would go on to join the International Congress of Mathemeticians in Zurich, becoming one of the most influential mathematician in the world of algebra. Noether lost her position at the university due to the influence of Germany's Nazi government. She then went to Bryn Mawr College in Pennsylvania.

During her time at Mathematical Institute of Erlangen, Nother made several contributions to algebraic invariants and number fields. She developed what is now known as Noether's theorem, which has become a staple theorem in theoretical physics and the calculus of variations. At the university of Gottingen, she made further contributions to theoretical physics when she published her paper *Theory of Ideals in Ring Domains* in 1921, which led to mathematicians describing objects that satisfy an ascending or descending chain conditions as Noetherian. She published several papers on noncommutative algebra and hypercomplex numbers. Noether would also inspire several mathematicians and was credited in numerous papers by other mathematicians that covered a wide range of fields.

Source: https://scientificwomen.net/women/noether-emmy-75