Fall 2019 COT 3100 Section 2 Fall Homework #5 Solutions

1) Use mathematical induction on n to prove the following assertion for all positive integers n:

$$\sum_{i=1}^{n}\frac{2}{i(i+2)}=\frac{3}{2}-\frac{(2n+3)}{n^2+3n+2}.$$

Proof

(1) Base case: n = 1.

$$\sum_{i=1}^{1}\frac{2}{i(i+2)}=\frac{2}{1\times(1+2)}=\frac{2}{3}$$

$$\frac{3}{2}-\frac{(2\times 1+3)}{1^2+3\times 1+2}=\frac{3}{2}-\frac{5}{6}=\frac{2}{3}$$

(2) Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$$\sum_{i=1}^{k}\frac{2}{i(i+2)}=\frac{3}{2}-\frac{(2k+3)}{k^2+3k+2}.$$

(3) Inductive step: Prove for n = k + 1 that

$$\sum_{i=1}^{k+1}\frac{2}{i(i+2)}=\frac{3}{2}-\frac{(2(k+1)+3)}{(k+1)^2+3(k+1)+2}$$

$$\sum_{i=1}^{k+1}\frac{2}{i(i+2)}=\sum_{i=1}^{k}\frac{2}{i(i+2)}+\frac{2}{(k+1)(k+1+2)}$$

$$=\frac{3}{2}-\frac{(2k+3)}{k^2+3k+2}+\frac{2}{(k+1)(k+1+2)}\quad \text{Using the induction hypothesis}$$

$$=\frac{3}{2}-\frac{(2k+3)}{(k+1)(k+2)}+\frac{2}{(k+1)(k+1+2)}$$

$$=\frac{3}{2}-\frac{(2k+3)(k+1+2)}{(k+1)(k+2)(k+1+2)}+\frac{2(k+2)}{(k+1)(k+2)(k+1+2)}$$

$$=\frac{3}{2}-\left(\frac{(2k+3)(k+1+2)}{(k+1)(k+2)(k+1+2)}-\frac{2(k+2)}{(k+1)(k+2)(k+1+2)}\right)$$

$$=\frac{3}{2}-\frac{(2k+3)(k+1+2)-2(k+2)}{(k+1)(k+2)(k+1+2)}$$

$$=\frac{3}{2}-\frac{(2k+3)(k+3)-2(k+2)}{(k+1)(k+2)(k+1+2)}$$

$$=\frac{3}{2}-\frac{2k^2+6k+3k+9-2k-4}{(k+1)(k+2)(k+1+2)}$$

$$=\frac{3}{2}-\frac{2k^2+7k+5}{(k+1)(k+2)(k+1+2)}$$

$$= \frac{3}{2} - \frac{(2k+5)(k+1)}{(k+1)(k+2)(k+1+2)}$$
$$= \frac{3}{2} - \frac{(2k+5)}{(k+2)(k+1+2)}$$
$$= \frac{3}{2} - \frac{2(k+1)+3}{(k+1+1)(k+1+2)}$$
$$= \frac{3}{2} - \frac{2(k+1)+3}{(k+1)^2+3(k+1)+2}$$

Based on the logic of mathematical induction, this proves that the original assertion holds for all positive integers n.

2) Define the sequence t_n for all non-negative integers n as follows:

$t_0 = 4$, $t_1 = 13$, $t_n = 5t_{n-1} - 6t_{n-2}$, for all integers $n \geq 2$.

Using strong induction on n, prove for all non-negative integers that $t_n = 5(3^n) - 2^n$. (Note: please use 2 base cases.)

Proof

(1) Base Cases:

n = 0, $t_2 = 5 \times (3^0) - 2^0 = 4$

n = 1, $t_3 = 5 \times (3^1) - 2^1 = 13$

From the 2 base cases, we can see $t_n = 5(3^n) - 2^n$ is true for both n = 0 and n = 1.

(2) Inductive Hypothesis: Assume for all $n \leq k$ (k is an arbitrary non-negative integer) that

$t_k = 5(3^k) - 2^k$

(3) Inductive Step: Prove for n = k+1

$t_{k+1} = 5(3^{k+1}) - 2^{k+1}$

Since $t_n = 5t_{n-1} - 6t_{n-2}$, we can see that:

$t_{k+1} = 5t_k - 6t_{k-1}$

$=5(5(3^k) - 2^k) - 6(5(3^{k-1}) - 2^{k-1})$ Using the induction hypothesis

$=25\times3^k - 5\times2^k - 2\times3\times5\times3^{k-1} + 2\times3\times2^{k-1}$

$=25\times3^k - 5\times2^k - 2\times5\times3^k + 3\times2^k$

$=25\times3^k - 5\times2^k - 10\times3^k + 3\times2^k$

$=25\times3^k - 10\times3^k - 5\times2^k + 3\times2^k$

$= (25 - 10)\times3^k - (5 - 3)\times2^k$

$=15\times3^k - 2\times2^k$

$=5\times3^{k+1} - 2^{k+1}$

Based on the logic of mathematical induction, this proves that the original assertion holds for all non-negative integers n.

3) For all positive integers n, define the n[th] Harmonic number, H_n, as follows: $H_n = \sum_{i=1}^{n} \frac{1}{i}$. For all positive integer, using induction on n, prove that $\sum_{i=1}^{n} H_i = (n+1)H_n - n$.

Proof

(1) Base case: n = 1

$H_1 = \sum_{i=1}^{1} \frac{1}{i} = 1$

LHS = $\sum_{i=1}^{1} H_i = 1$

RHS = (1 + 1) H_1 - 1 = 1

(2) Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$$\sum_{i=1}^{k} H_i = (k+1)H_k - k$$

(3) Inductive step: Prove for n = k+1 that

$$\sum_{i=1}^{k+1} H_i = (k+1+1)H_{k+1} - (k+1)$$

$$\sum_{i=1}^{k+1} H_i = \sum_{i=1}^{k} H_i + H_{k+1}$$

$= (k+1)H_k - k + H_{k+1}$ Use the induction hypothesis

$= (k + 1) \sum_{i=1}^{k} \frac{1}{i} - k + H_{k+1}$

$= (k + 1)(\sum_{i=1}^{k} \frac{1}{i} + \frac{1}{k+1} - \frac{1}{k+1}) - k + H_{k+1}$

$= (k + 1)(\sum_{i=1}^{k+1} \frac{1}{i} - \frac{1}{k+1}) - k + H_{k+1}$

$= (k + 1) \sum_{i=1}^{k+1} \frac{1}{i} - 1 - k + H_{k+1}$

$= (k + 1) H_{k+1} - 1 - k + H_{k+1}$

$= (k + 1 + 1) H_{k+1} - (k + 1)$

Based on the logic of mathematical induction, this proves that the given assertion is true for all Integers n > 0.

4) Using proof by induction on n, prove that $8 \mid (3^{2n+1} + 5^{2n+1})$ for all non-negative integers n.

Proof

(1) Base case: n = 0

$(3^{2(0)+1} + 5^{2(0)+1}) = 3 + 5 = 8$

$8 = 8 \times 1$

(2) Inductive hypothesis: Assume for an arbitrary non-negative integer n = k that

$$8 \mid (3^{2k+1} + 5^{2k+1})$$

(3) Inductive step: Prove for n = k+1 that

$$8 \mid (3^{2(k+1)+1} + 5^{2(k+1)+1})$$

From the inductive hypothesis, we can get: $(3^{2k+1} + 5^{2k+1}) = 8c$ (c is a non-negative integer).

$$\begin{aligned}3^{2(k+1)+1} + 5^{2(k+1)+1} &= 3^{2k+2+1} + 5^{2k+2+1} \\ &= 9 \times 3^{2k+1} + 25 \times 5^{2k+1} \\ &= 9 \times (8c - 5^{2k+}) + 25 \times 5^{2k+1} \\ &= 9 \times 8c - 9 \times 5^{2k+} + 25 \times 5^{2k+1} \\ &= 9 \times 8c + 16 \times 5^{2k+1} \\ &= 8(9c + 2 \times 5^{2k+1})\end{aligned}$$

Since k and c both are non-negative integers, $(9c + 2 \times 5^{2k+1})$ is an integer. It follows that the original assertion holds for all non-negative integers n.

5) Give a summary of the life and mathematical contributions of Evariste Galois. Please aim for a length of roughly 200 - 400 words. ***<u>Your summary must be typed.</u>*** Please state the sources you used in writing your summary.

Galois was born in 1811 and his father was a prominent Republican, which came into play later in his life. Evariste Galois made contributions to nearly all branches of mathematics during his short life. For many times, his discoveries failed to be published due to various troubles.

He is one of the founders of the branch of algebra known as group theory. He developed the concept of normal subgroup and Galois field. He made basic studies of linear groups over finite fields. He constructed the general linear group over a prime field and projective special linear group.

He developed Galois theory which is his most significant contribution to mathematics. He related the algebraic solution to a polynomial equation to the structure of a group of permutations associated with the roots of the polynomial. He found if an equation's Galois group has a series of subgroups, it could be solved in radicals.

For the analysis mathematics, Evariste made some contributions to the theory of Abelian integrals and continued fractions. He classified Abelian integrals into three categories.

Evariste also did important work on continued fractions. He proved that the regular continued fraction which represents a quadratic surd ζ is purely periodic if and only if ζ is a reduced surd. Evariste Galois's work had a significant overlap with the work of Niels Henrik Abel another mathematician.

On a historical note, Galois was extremely interested in politics. In 1827, Charles X had suffered a political setback, but in 1830, he staged a coup d'etat. As this was occurring, Galois's school, the Ecole Normale was locked. Rather than deal with this, Galois quit school and joined the Republican artillery unit of the National Guard. Based on this association, Galois was put in jail (and eventually released) multiple times. Naturally, between being in jail and his various political activities, he has less time to work on mathematics. Nonetheless, he still made some contributions during this time. Unfortunately, Galois died in a duel that took place on May 30, 1832. The circumstances around the duel are quite mysterious, but there's some reasonable evidence that the duel was over a woman, but there's no credible account of who the killer was and the written trail of evidence is fairly scant. It's shocking that someone who died at such a young age (20 years ago), was able to start a new branch of mathematics. Incidentally, much of modern public key cryptography works due to properties of Group Theory.

Sources: https://en.wikipedia.org/wiki/%C3%89variste_Galois