Fall 2019 COT 3100 Section 2 Fall Homework #4 Solutions

1) Find all integer solutions to the equation 1141x + 406y = 28.

Solution

We can divide both sides by 7 to simplify.

163x + 58y = 4

Extended Euclid's algorithm

163 = 2(58) + 47	47 = 163 – 2(58)
58 = 1(47) + 11	11 = 58 – 1(47)
47 = 4(11) + 3	3 = 47 – 4(11)
11 = 3(3) + 2	2 = 11 – 3(3)
3 = 1(2) + 1	1 = 3 – 1(2)

Substituting

1 = 3 – 1(2)
1 = 3 – 1(11 – 3(3))
1 = 3 - 11 + 3(3)
1 = -11 + 4(3)
1 = -11 + 4(47 – 4(11))
1 = -11 + 4(47) – 16(11)
1 = 4(47) – 17(11)
1 = 4(47) – 17(58 – 1(47))
1 = 4(47) – 17(58) + 17(47)
1 = -17(58) + 21(47)
1 = -17(58) + 21(163 – 2(58))
1 = -17(58) + 21(163) – 42(58)
1 = 163(21) + 58(-59)

Multiply both sides by 4

4*1 = 163(4*21) + 58(4*-59)
4 = 163(84) + 58(-236)
4 = 163(84 - 58) + 58(-236 + 163), for a simpler initial x and y
4 = 163(26) +58(-73)

Final Answer:

$$\{(\mathbf{x}, \mathbf{y}) \mid \forall \boldsymbol{c} \in \boldsymbol{Z}\ (\boldsymbol{x} = \mathbf{26} - \mathbf{58}\boldsymbol{c}, \boldsymbol{y} = -\mathbf{73} + \mathbf{163}\boldsymbol{c})\}$$

2) (a)Find all integer solutions to the equation 105x + 83y = 1.

(b) Find all integer solutions to the equation 105x + 83y = 8.

(c) Find 83^{-1} mod 105. (Note: Answer must be in between 0 and 104, inclusive.)

Solution

a)

Extended Euclid's algorithm

$105 = 1(83) + 22$	$22 = 105 - 1(83)$
$83 = 3(22) + 17$	$17 = 83 - 3(22)$
$22 = 1(17) + 5$	$5 = 22 - 1(17)$
$17 = 3(5) + 2$	$2 = 17 - 3(5)$
$5 = 2(2) + 1$	$1 = 5 - 2(2)$

Substituting

$1 = 5 - 2(2)$

$1 = 5 - 2(17 - 3(5))$

$1 = 5 - 2(17) + 6(5)$

$1 = -2(17) + 7(5)$

$1 = -2(17) + 7(22 - 1(17))$

$1 = -2(17) + 7(22) - 7(17)$

$1 = 7(22) - 9(17)$

$1 = 7(22) - 9(83 - 3(22))$

$1 = 7(22) - 9(83) + 27(22)$

$1 = -9(83) + 34(22)$

$1 = -9(83) + 34(105 - 1(83))$

$1 = -9(83) + 34(105) - 34(83)$

$1 = 105(34) + 83(-43)$

Final answer:

$$\{(\mathbf{x}, \mathbf{y}) \mid \forall \boldsymbol{c} \in \boldsymbol{Z}\ (\boldsymbol{x} = \mathbf{34} - \mathbf{83}\boldsymbol{c}, \boldsymbol{y} = -\mathbf{43} + \mathbf{105}\boldsymbol{c})\}$$

b)

Multiply both sides of final equation from part (a) by 8

8*1 = 8*(105(34) + 83(-43))

8 = 105(8*34) + 83(8*-43)

8 = 105(272) + 83(-344)

8 = 105(272-(83*3)) + 83(-344-(105*3)), for a simpler initial x and y

8 = 105(23) + 83(-29)

Final answer:

$\{(\mathbf{x}, \mathbf{y}) \mid \forall \boldsymbol{c} \in \boldsymbol{Z}\ (\boldsymbol{x} = \mathbf{23} - \mathbf{83}\boldsymbol{c}, \boldsymbol{y} = -\mathbf{29} + \mathbf{105}\boldsymbol{c})\}$

c)

Use the final equation from part (a):

105(34) + 83(-43) = 1

Use mod 105 to get:

0 + 83(-43) = 1 (mod 105)

So, $83^{-1} \equiv -43 \pmod{105}$

Must get in between 0 and 104

-43 ≡ 62 (mod 105)

Final answer:

62

3) Let x and y be integers such that 15 | (3x + 4y). Prove that 15 | (12x + y).

Solution

For some integers a, b, and c:

15ax + 15by + c(3x + 4y) = 12x + y

15ax + 15by + 3cx + 4cy = 12x + y

(15a+3c)x + (15b+4c)y = 12x + y

15a + 3c = 12

15b + 4c = 1

Use mod 15 to get rid of a and b variables

3c ≡ 12 (mod 15)

4c ≡ 1 (mod 15)

Multiply both sides of the second equation by 4 to obtain

4c ≡ 1 (mod 15)

4(4c) ≡ 4 (mod 15)

c ≡ 4 (mod 15)

Note: We chose 4 since $4^{-1} \equiv 4 \pmod{15}$. Formally, we would do the Extended Euclidean Algorithm to determine this, but it's fairly easy to eyeball.

Plug in c to find a and b

15a + 3(4) = 12	15b + 4(4) = 1
15a + 12 = 12	15b + 16 = 1
15a = 0	15b = -15
a = 0	b = -1

Plug a,b, and c back into original equation

15(0)x +15(-1)y + 4(3x + 4y) = 12x + y

Since 15 | (3x + 4y), $\exists d \in Z | (3x + 4y) = 15d$

Thus,

12x + y = 15(0)x +15(-1)y + 4(3x + 4y) = 15(0)x +15(-1)y + 4(15d) = 15(-y+4d)

Since both y and d are integers, it follows that -y+4d is as well. Thus, we can conclude that 15 | (12x + y), as desired.

4) Let a = $2^4 3^2 5^6 7^8$, b = $2^2 3^9 5^4 11^5$, and c = $2^5 3^7 5^3 11^2$. Determine, in prime factorized form, both gcd(a, b, c) and lcm(a, b, c).

Solution
For gcd keep min powers of prime factors between a, b, and c:
$\mathbf{2^2 3^2 5^3}$

For lcm keep max powers of prime factors between a, b, and c:
$\mathbf{2^5 3^9 5^6 7^8 11^5}$

5) For the numbers a, b and c listed in problem 4, determine the number of divisors each of those numbers has.

Solution
a) (4+1)(2+1)(6+1)(8+1) = 5*3*7*9 = **945 divisors**
b) (2+1)(9+1)(4+1)(5+1) = 3*10*5*6 = **900 divisors**
c) (5+1)(7+1)(3+1)(2+1) = 6*8*4*3 = **576 divisors**

6) How many zeroes are at the end of $\frac{1000!}{500!500!}$?

Solution
Find the number of zeroes at the end 1000! and 500! Separately

# of tailing zeroes of 1000! = $\frac{1000}{5}+\frac{1000}{25}+\frac{1000}{125}+\frac{1000}{625}=200+40+8+1=249$

# of tailing zeroes of 500! = $\frac{500}{5}+\frac{500}{25}+\frac{500}{125}=100+20+4=124$

Use powers of 10 to represent # of zeroes

$$\frac{10^{249}}{10^{124}*10^{124}}=\frac{10^{249}}{10^{124+124}}=\frac{10^{249}}{10^{248}}=10^{249-248}=10^1$$

Final answer:
1 zero

7) The following was proved in class: if p is a prime number and a and b are positive integers such that p | ab, then either p | a or p | b. Show that this statement is false for composite numbers. Namely, disprove the following statement: if n is a composite number and a and b are positive integers such that n | ab, then either n | a or n | b.

Solution

Consider the counterexample when $p = 6$, $a = 3$, and $b = 2$.
$a*b = 3*2 = 6$
$6|6$, but $6\nmid 3$ and $6\nmid 2$
Therefore, the given statement is **false.**

8) In honor of my new license plate (image below), please give a summary of the life and mathematical contributions of Leonard Euler. Please aim for a length of roughly 200 - 400 words. ***<u>Your summary must be typed.</u>*** Please state the sources you used in writing your summary.

Leonhard Euler was born in April 15 in 1707 in Basel Switzerland. He moved to St. Petersburg in 1727 and become an associate at the St. Petersburg Academy of Sciences and become of the chair of mathematics in 1733. . In the mid-1740s, Euler was appointed the mathematics director of the newly created Berlin Academy of Science and Beaux Arts, taking on a variety of management roles as well becoming head of the organization itself for a time starting in 1759. Not appointed president proper of the academy by King Frederick II, Euler received patronage from Catherine II and in 1766 returned to Russia to head the St. Petersburg Academy.

He developed the common concept of functions in mathematical analysis. He also advanced the use of infinitesimals and infinite quantities. He also advanced the field of geometry and trigonometry. He also discovered the imaginary logarithms of negative numbers and showed that each complex number has an infinite number of logarithms.

He also made several advancements in the field of calculus. He created a method of integration that would determine the work done by a force. He also introduced a lot of new notation such sigma for sum. He also has e, the base of natural logarithms, named after him, as it is called Euler's constant. He standardized the use of i to denote the square root of negative one. He helped to standardize the use of the Greek letter pi as the symbol for the ratio of a circle's circumference. He even standardized the use of a, b, c for the sides of a triangle. Euler also greatly contributed to the field of number theory, especially when he discovered the law of quadratic reciprocity, which is now essential.

Some neat results due to Euler's work are:

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

For all planar graphs, V - E + F = 2, where V is the number of vertices, E is the number of edges and F is the number of faces.

A connected graph has an Euler Circuit if and only if the degree of each vertex in the graph is even.

For all positive integers a and n with gcd(a, n) = 1, $a^{\phi(n)} \equiv 1 \ (mod\ n)$, where φ(n) is the Euler Phi function, equal to the number of integers from the set {1, 2, 3, …, n} that are relatively prime to n.

Sources: https://www.storyofmathematics.com/18th_euler.html
<u>Euler: The Master of Us All</u> by William Dunham