<u>Fall 2019 COT 3100 Section 2 Homework #3 Solutions</u>

1) Let n be a positive odd integer. Prove that $n^4 \equiv 1 \pmod{16}$. You may use any of the mod rules provided in class or the recommended course text book. (In addition, if you end up getting an expression of the form a(3a+1) somewhere in your work where a is an integer, separately end up proving that this expression is always even and use that result to complete your proof.)

<u>Solution</u>
Since n is an odd integer, there exists an integer a such that n = (2a + 1).

We want to prove $(2a + 1)^4 \equiv 1 \pmod{16}$.

$n^2 = (2a + 1)^4 = (4a^2 + 4a + 1)^2 = (4a(a + 1) + 1)^2$

In class it was proven that if a is an integer, then a(a + 1) is even. Thus, there exists an integer d such that a(a+1) = 2d, substitute accordingly:

$(4a(a + 1) + 1)^2 =$
$(4(2d) + 1)^2 =$
$(8d + 1)^2 =$
$64d^2 + 16d + 1 =$
$16(4d^2 + d) + 1 \equiv 1 \pmod{16}$, since $16 \equiv 0 \pmod{16}$, and d is an integer.

2) Convert the following values from the bases indicated to base 10:

i) 3214_7
ii) FCE_{16}
iii) 35142_8
iv) 20031_4
v) 110001010011_2

<u>Solution</u>
i) $3 * 7^3 + 2 * 7^2 + 1 * 7 + 4 =$ **<u>1138</u>**
ii) F = 15 C = 12 E = 14
$15 * 16^2 + 12 * 16 + 14 =$ **<u>4046</u>**
iii) $3 * 8^4 + 5 * 8^3 + 1 * 8^2 + 4 * 8 + 2 =$ **<u>14946</u>**
iv) $2 * 4^4 + 0 * 4^3 + 0 * 4^2 + 3 * 4^1 + 1 * 4^0 =$ **<u>525</u>**
v) 110001010011
$1 * 2^{11} + 1 * 2^{10} + 1 * 2^6 + 1 * 2^4 + 1 * 2^1 + 1 * 2^0 = 4096 + 2048 + 128 + 32 + 2 + 1 =$ **<u>3155</u>**

3) Convert the following base 10 values to the bases indicated:

i) 83111 to base 12
ii) 23650 to base 16
iii) 831 to base 2
iv) 3426 to base 7
v) 4319 to base 8

Solution
Using | to indicate division

i)

83111 | 12	6925 R 11
6925 | 12	577 R 1
577 | 12	48 R 1
48 | 12	4 R 0
4 | 12	0 R 4

83111 = $\underline{\mathbf{4011B_{12}}}$

ii)

23650 | 16	1478 R 2
1478 | 16	92 R 6
92 | 16	5 R 12
5 | 16	0 R 5

23650 = $\underline{\mathbf{5C62_{16}}}$

iii)

831 | 2	415 R 1
415 | 2	207 R 1
207 | 2	103 R 1
103 | 2	51 R 1
51 | 2	25 R 1
25 | 2	12 R 1
12 | 2	6 R 0
6 | 2	3 R 0
3 | 2	1 R 1
1 | 2	0 R 1

831 = $\underline{\mathbf{1100111111_{2}}}$

iv)

3426 | 7	489 R 3
489 | 7	69 R 6
69 | 7	9 R 6
9 | 7	1 R 2
1 | 7	0 R 1

3426 = $\underline{\mathbf{12663_{7}}}$

v)
4319 | 8 539 R 7
539 | 8 67 R 3
67 | 8 8 R 3
8 | 8 1 R 0
1 | 8 0 R 1
4319 = $\underline{\mathbf{10337_8}}$

4) Jessica and Martin start riding their bicycles towards each other at 2 pm. At 2 pm, they are 25 miles apart. Jessica rides her bike at a constant rate of 15 miles per hour and Martin rides his at a constant rate of 10 miles per hour. At 2 pm, a bird starts flying towards Martin. As soon as the bird gets to Martin, it turns back around and flies towards Jessica, and continues going back and forth until Jessica and Martin meet. The bird travels at a constant rate of 45 miles per hour. How far does the bird fly from the time it starts until Jessica and Martin meet? Assume that it takes the bird no time to turn around and fly the other direction

Solution
Since Jessica and Martin are traveling towards the same goal we can combine their speeds.

J + M = 25 m/h and we can represent speed as distance over time, so to find how long they are travelling we can represent it as J + M = D/t where D is the total distance of 25 miles and solve for t.

t = 1 hour, until the two meet.

Since the bird travels constantly between them we can use the same formula to find the distance for the bird. 45m/h = D/1 hour.
D = **45 miles.**

5) A common divisibility rule is that a positive integer $n = d_k d_{k-1} \ldots d_0$, (where each d_i represents a single digit of n), is divisible by 9 if and only if $d_k + d_{k-1} + \ldots + d_0$ is divisible by 9. **Discover a somewhat similar divisibility rule for 11 and rigorously prove this via mod rules.**

Solution
Looking at a number with digits $a_k a_{k-1} \ldots a_0$ we can represent this as $d_k 10^k + d_{k-1} 10^{k-1} \ldots d_0 10^0$. Now, consider the value of this number mod 11, keeping in mind that $10 \equiv -1 \pmod{11}$:

$$d_k 10^k + d_{k-1} 10^{k-1} \ldots d_0 10^0 \equiv d_k(-1)^k + d_{k-1}(-1)^{k-1} \ldots d_0(-1)^0 \pmod{11}$$
$$\equiv d_0 - d_1 + d_2 - d_3 \ldots +(-1)^k d_k \pmod{11}$$

This, means, that if some number is divisible by 11, then the latter value listed will be equivalent to 0 (mod 11). Thus, a divisibility rule for 11 mathematically is as follows:

If n is divisible by 11, then,

$$d_0 - d_1 + d_2 - d_3 \ldots +(-1)^k d_k \equiv 0 \pmod{11}$$

It's fairly easy to see that this is an if and only if. Now, let's work with the equation above:

$d_0 - d_1 + d_2 - d_3 \ldots +(-1)^k d_k \equiv 0 \pmod{11} \leftrightarrow$
$d_0 + d_2 + d_4 + \ldots \equiv d_1 + d_3 + d_5 + \ldots \pmod{11}$

Thus, an integer n is divisible by 11 if and only if the sum of the digits at the even places and the sum of the digits at the odd places are both equivalent mod 11.

Or, another way to say it is, take the alternating sum of digits (add the least significant, subtract the tens digit, add the 100s digit, etc.) and this alternating sum must be divisible by 11. Note that we can start the alternating sum at the most significant or least significant digit if we are only testing for divisibility and don't care about non-zero mod values.

Consider the following integer:

4423264152133

The alternating digit sum from the left is 3 - 3 + 1 - 2 + 5 - 1 + 4 - 6 - 2 + 3 - 2 + 4 - 4 = 0, so this number is divisible by 11.

96988672 is also divisible by 11 because the alternative digit sum from the left is

2 - 7 + 6 - 8 + 8 - 9 + 6 - 9 = -11, and this is divisible by 11.

6) Let x and y be integers such that 17 | (3x + 5y). Prove that 17 | (8x + 19y).

Solution

$8x + 19y = 17x - 9x + 34y - 15y$
$= 17(x + 2y) -3(3x+5y)$

Since 17 | (3x + 5y), there exists an integer c such that 3x + 5y = 17c. Substituting, we get:

$= 17(x + 2y) - 3(17c)$
$= 17(x + 2y - 3c)$

Since, x, y and c are integers, x + 2y - 3c is an integer, thus, proving that 17 | (8x + 19y).

We can solve this without "guessing and checking" as follows:

We know that our goal is to find some integer c such that

$$8x + 19y \equiv c(3x + 5y) \pmod{17}$$
$$8x + 19y \equiv 3cx + 5cy \pmod{17}$$

Equating coefficients, our goal is to find c such that $8 \equiv 3c \pmod{17}$ and $19 \equiv 5c \pmod{17}$.

Multiply the first equation through by 3^{-1} mod 17. Let's find this value via the Extended Euclidean Algorithm:

17 = 5 x 3 + 2
3 = 1 x 2 + 1
3 - 1 x 2 = 1
3 - 1(17 - 5 x 3) = 1
3 - 1x17 + 5x3 = 1
6x3 - 1x17 = 1
Taking this equation mod 17, we find:
6x3 ≡ 1 (mod 17) so $3^{-1} \equiv 6 \pmod{17}$.
3c ≡ 8 (mod 17)
6(3c) ≡ 6(8) (mod 17)
c ≡ 48 ≡ -3 (mod 17)

Thus, we find that -3(3x + 5y) = -9x -15y. Quickly we can see that 8 - (-9) = 17 and 19 - (-15) = 34 and that both of these are multiples of 17, so this gives us the necessary breakdown to expression 8x + 19y without guessing. The rest of the proof follows as shown in the first solution.

7) Give a summary of the life and mathematical contributions of Carl Friedrich Gauss. Please aim for a length of roughly 200 - 400 words. ***Your summary must be typed.*** Please state the sources you used in writing your summary.

Sample Write Up
Carl Friedrich Gauss is known as one of the world's best mathematicians. Born in present day Germany in 1777, Gauss showed talent in mathematics very early on. Anecdotally, he is given credit for discovering how to add the first hundred integers at the age of 8. By the age of 21, he had produced a ground breaking work on Number Theory, Disquisitiones Arithmeticae. One of Gauss's early triumphs was determining exactly which regular polygons could and could not be constructed with a compass and a straight edge. Though this appears to be a geometry problem, the ultimate solution heavily involves number theory as he found that the construction is possible if and only if the number of sides is a product of distinct Fermat primes and a power of 2. (Note: A Fermat prime is a prime number of the form $2^k + 1$, where k is a positive integer. It turns out that number of this form may be prime only if k itself is a power of 2. There are only 5 known Fermat primes.) Another notable result of Gauss's was determining an efficient method for calculating which values are quadratic residues. (A value q is a quadratic residue mod n if and only if there is an integer x such that $x^2 \equiv q \pmod{n}$.) The rule that is used to make this calculation is Gauss's Law of Quadratic Reciprocity. Later in Gauss's career, he studied fields other than mathematics. Outside of mathematics, he's best known for his contribution to magnetism with the law that is simply now known as Gauss's Law. The law states that, "the area integral of the electric field over any closed surface is equal to the net charge enclosed in the surface divided by the permittivity of space." Gauss's Law is one of four equations known as Maxwell's Equations, which are the key cornerstone to any physics course in electricity and magnetism.

Sources
https://en.wikipedia.org/wiki/Carl_Friedrich_Gauss
http://hyperphysics.phy-astr.gsu.edu/hbase/electric/gaulaw.html