COT3100 Homework - 2

Solutions

September 26, 2019

Problem 1. Prove that if n is an off integer, then $8 \mid (n^2 - 1)$. You may use a previously proved result from class that for any integer a, a(a+1) is an even integer.

Proof. n is odd, therefore, there exists some arbitrary integer k such that $n = 2k + 1$.
$n^2 - 1 = (2k - 1)^2 - 1$
$4k^2 - 4k + 1 - 1$
$4k^2 - 4k$
$4k(k - 1)$
Because for any integer a, a(a+1) is even, we know that for some arbitrary integer b,
$k(k - 1) = 2b$
$4k(k - 1) = 4(2b) = 8b$
Because b is an integer, and $(n^2 - 1) = 8b$ when n is odd, $8 \mid (n^2 - 1)$ when n is odd. □

Problem 2. Prove or disprove: If an integer n has three unique prime divisors, then it follows that the largest prime divisor of n is less than or equal to $\sqrt[3]{n}$

Proof. This is false. Consider the following counter-example: n = 606. The largest prime factor of this value is 101, but $\sqrt[3]{606} \sim 8.46$. Note that the statement would be true for the *smallest* prime divisor of n. □

Problem 3. A triangle has all integer side lengths and two of those sides have lengths 9 and 16. Consider the altitudes to the three sides. What is the largest possible value of the ratio of any of those two altitudes.

Proof. We can solve this problem using the standard formula for the area of a triangle and the triangle inequality. Let a be the length of one side of a triangle, let h_a be the altitude of the triangle to that side, and let A be the area of the triangle.

We have $2A = a$ x h_a. The triangle inequality simply states that the length of any two sides of a triangle sum up to a value greater than the length of the third side. Thus, for this problem, we see that the third side must be at least 8 units long (since it's an integer and 7 is too small since 7 + 9 = 16), and that the maximum length is 24 units long (since it's an integer and 25 is too long since 9 + 16 = 25.) Let two of the sides of the triangle be a, and b, with altitudes h_a, and h_b, respectively.

$2\text{A} = ah_a = bh_b$
Solving for the ratio of of altitudes we get:

$\frac{h_a}{h_b} = \frac{b}{a}$

It follows that if we want to maximize or minimize this ratio, we just want to maximize or minimize the ratio of the sides of the triangle. Notice that if $\frac{a}{b}$ is the maximum ratio, then $\frac{b}{a}$ would be the minimum ratio. Thus, we just need to try both the minumum and maximum side lengths for the triangle to see which of the two creates the largest ratio of sides. If we choose a third side length of 8, then the corresponding ratio of the largest to the smallest side is $\frac{16}{8}$. Alternatively, if we choose a third side length of 24, then the corresponding ratio of the largest to the smallest side is $\frac{24}{9}$. This is larger than $\frac{16}{8}$. It follows that the maximum possible ratio of two altitudes of the triangle is $\frac{8}{3}$, and the minimum possible ratio is $\frac{3}{8}$. □

Problem 4. Consider the two different numbers 327_b and $327_(b+1)$, where b is a positive integer 8 or greater. If the difference between these two numbers is 89, what is the value of b?

Proof. Using the given information, we have:
$3\text{b}^2 + 2b + 7 + 89 = 3(b+1)^2 + 2(b+1) + 7$
$3b^2 + 2b + 7 + 89 = 3b^2 + 6b + 3 + 2b + 2 + 7$
$89 = +6b + 3 + 2$
$84 = 6b$
$b = 14$ □

Problem 5. Let S = 2, 6, 8 ,9
Let T = 1, 2, 5
Find $S \cup T, S \cap T, S - T, P(S), P(T)$

Proof. $S \cup T = 1, 2, 6, 5, 8, 9$
$S \cap T = 2$
$S - T = 6, 8, 9$
$S \times T = (2,1), (2,2), (2,5), (6,1), (6,2), (6,5), (8,1), (8,2), (8,5), (9,1), (9,2), (9,5)$
$T \times S = (1,2), (1,6), (1,8), (1,9), (2,2), (2,6), (2,8), (2,9), (5,2), (5,6), (5,8), (5,9)$
$P(S) = \{\}, \{2\}, \{6\}, \{8\}, \{9\}, \{2,6\}, \{2,8\}, \{2,9\}, \{6,8\}, \{6,9\}, \{8,9\}, \{2,6,8\}, \{2,6,9\}, \{2,8,9\}, \{6,8,9\}$
$, \{2,6,8,9\}$
$P(T) = \{\}, \{1\}, \{2\}, \{5\}, \{1,2\}, \{1,5\}, \{2,5\}, \{1,2,5\}$

□

Problem 6. Use set laws to show the two following sets are equivalent
a) $\neg((\neg A \cap \neg B) \cup (C \cap \neg C)) \cup ((A \cup B) \cap (A \cap B))$
b) $A \cup B$

Proof. 1: $\neg((\neg A \cap \neg B) \cup (C \cap \neg C)) \cup ((A \cup B) \cap (A \cap B))$
2: $\neg((\neg A \cap \neg B) \cup (\varnothing)) \cup ((A \cup B) \cap (A \cap B))$ - Inverse
3: $\neg((\neg A \cap \neg B)) \cup ((A \cup B) \cap (A \cap B))$ - Identity
4: $(\neg\neg A \cup \neg\neg B) \cup ((A \cup B) \cap (A \cap B))$ - DeMorgan's
5: $(A \cup B) \cup ((A \cup B) \cap (A \cap B))$ - Double Negation
6: $(A \cup B)$ - Absorption

□

Problem 7. Give a summary of the life and work of mathematician Maryam Mirzakhani. Please aim for a length of 200 to 400 words. Please state the sources you used in writing your summary.

Maryam Mirzahkani was a mathematician, who is best known for being the first woman and first Iranian to be honored with a Fields Medal for her work in the geometry of Riemann surfaces and their moduli spaces. She showed an aptitude for mathematics at an early age, earning a Gold Medal in both the Iranian National Olympiad and the International Mathematical Olympiad (IMO) in 1994 and 1995. In the latter year, she earned a perfect score at the IMO. After high school, Mirzahkani attended Sharif University of Technology, followed by graduate school at Harvard University, where she earned her Ph.D. in mathematics in 2004.

After she finished her studies, Dr. Mirzahhani worked at the Clay Mathematics Institute, Princeton University and Stanford University. Her Ph.D. involved solving a previously unsolved problem for coungint the number of simple closed geodescics of length less than L, in terms of L. She showed that this is a polynomial and bounded the polynomial based on the genus of the structure.

She continued her work in this area over the course of the next several years, and was able to prove a conjecture about William Thurston's earthquake flow. In 2014, in collaboration with Alex Eskin and Amir Mohammadi, she proved the regularity of complex geodesics and their closures in modular space. It was for body of this work that she received the Fields Medal in 2014. Unfortunately, Dr. Mirzahkani was diagnosed with breast cancer in 2013 and died on July 14, 2017, at the age of 40.

Source: https://en.wikipedia.org/wiki/Maryam_Mirzakhani