Fall 2018 COT 3100 Section 2 Homework #1 Solutions

1) Fill out the following truth table:

Solution

p	q	r	$\bar{p}$	$r \vee \bar{p}$	$q \wedge (r \vee \bar{p})$	$p \oplus (q \wedge (r \vee \bar{p}))$
F	F	F	**T**	**T**	**F**	**F**
F	F	T	**T**	**T**	**F**	**F**
F	T	F	**T**	**T**	**T**	**T**
F	T	T	**T**	**T**	**T**	**T**
T	F	F	**F**	**F**	**F**	**T**
T	F	T	**F**	**T**	**F**	**T**
T	T	F	**F**	**F**	**F**	**T**
T	T	T	**F**	**T**	**T**	**F**

2) Use the laws of logic to show that the following expression is a tautology:

$$\left(p \wedge (p \vee (r \wedge s))\right) \vee \left(\bar{p} \wedge ((q \vee \bar{s}) \vee s)\right)$$

Solution

$\left(p \wedge ((p \vee r) \wedge (p \vee s))\right) \vee \left(\bar{p} \wedge ((q \vee \bar{s}) \vee s)\right)$	Distributive law
$\left((p \wedge (p \vee r)) \wedge (p \vee s)\right) \vee \left(\bar{p} \wedge ((q \vee \bar{s}) \vee s)\right)$	Associative law
$(p \wedge (p \vee s)) \vee \left(\bar{p} \wedge ((q \vee \bar{s}) \vee s)\right)$	Absorption law
$p \vee \left(\bar{p} \wedge ((q \vee \bar{s}) \vee s)\right)$	Absorption law
$p \vee \left(\bar{p} \wedge (q \vee (\bar{s} \vee s))\right)$	Associative law
$p \vee (\bar{p} \wedge (q \vee T))$	Inverse law
$p \vee (\bar{p} \wedge T)$	Domination law
$p \vee \bar{p}$	Identity law
T	Inverse law

3) Let r_1 and r_2 be the roots of the quadratic equation $x^2 - 12x + 9 = 0$. Determine the value of $r_1^2 + r_2^2$. (Hint: In the first recitation the relationship between the coefficients in a quadratic and the sum and product of the roots of the quadratic was proven. Use this relationship to solve the problem without finding either root.)

Solution

$$x^2 - 12x + 9 = 0$$
$$x^2 - 12x + 9 = (x - r_1)(x - r_2) = 0$$
$$x^2 - 12x + 9 = x^2 - (r_1 + r_2)x + r_1r_2 = 0$$

We can see that 12 *corresponds to* $(r_1 + r_2)$ *and* 9 *to* r_1r_2 *and thus*:

$$r_1 + r_2 = 12 \qquad r_1r_2 = 9$$
$$(r_1 + r_2)^2 = 12^2$$
$$r_1^2 + 2r_1r_2 + r_2^2 = 144$$
$$r_1^2 + 2(9) + r_2^2 = 144 \qquad \textit{we substitute } r_1r_2 \textit{ with } 9$$
$$r_1^2 + r_2^2 = 144 - 18$$
$$r_1^2 + r_2^2 = \mathbf{126}$$

4) Sarah averages 30 miles per hour driving from Orlando to Tampa. What must her average speed be on the return trip if the average speed for the whole round trip was 40 miles an hour? In your work, prove that the answer holds no matter what the distance between Orlando and Tampa is. (Namely, the answer would be true for any two arbitrarily chosen cities.)

Solution

D is distance from Orlando to Tampa, t_1 is time it took Sarah to drive from Orlando to Tampa and t_2 is the time Sarah drove from Tampa to Orlando

$$\frac{D}{t_1} = 30 \qquad \frac{2D}{t_2 + t_1} = 40 \qquad \textit{We want to find } \frac{D}{t_2}$$
$$D = 30t_1$$
$$\frac{2(30t_1)}{t_1 + t_2} = 40 \qquad \textit{We replace } D \textit{ with } 30t_1$$
$$60t_1 = 40(t_1 + t_2)$$
$$60t_1 = 40t_1 + 40t_2$$
$$20t_1 = 40t_2$$
$$\frac{t_1}{2} = t_2$$
$$\frac{D}{t_2} = \frac{D}{\frac{t_1}{2}} \qquad \textit{We replace } t_2 \textit{ by } \frac{t_1}{2}$$
$$\frac{D}{t_2} = \frac{2D}{t_1} = 2(30) = 60 \qquad \textit{We replace } \frac{D}{t_1} \textit{ by } 30$$

The answer is $\mathbf{60\ mph}$.

5) Show the result of the following bitwise operations. Please show your work (as it's trivial to type these into any programming language and get the result.) In particular, show the conversion of the values to binary, followed by your bit by bit calculation. If you don't know how to do the binary conversion using the proper algorithm by repeatedly dividing by 2, just work it out intuitively adding up powers of 2. (Note: r is the remainder in the work below.)

$(a)132|85$ (b) 217&95 (c) 88^221

Solution

(a) $132|85$

$$\frac{132}{2} = 66\,, r = 0$$
$$\frac{66}{2} = 33\,, r = 0$$
$$\frac{33}{2} = 16\,, r = 1$$
$$\frac{16}{2} = 8\,, r = 0$$
$$\frac{8}{2} = 4\,, r = 0$$
$$\frac{4}{2} = 2\,, r = 0$$
$$\frac{2}{2} = 1\,, r = 0$$
$$\frac{1}{2} = 0\,, r = 1$$

$$\frac{85}{2} = 42\,, r = 1$$
$$\frac{42}{2} = 21\,, r = 0$$
$$\frac{21}{2} = 10\,, r = 1$$
$$\frac{10}{2} = 5\,, r = 0$$
$$\frac{5}{2} = 2\,, r = 1$$
$$\frac{2}{2} = 1\,, r = 0$$
$$\frac{1}{2} = 0\,, r = 1$$

$132 \textit{ in binary is } 10000100 \textit{ and } 85 \textit{ in binary is } 1010101$

132	85	132|85
1	0	1
0	1	1
0	0	0
0	1	1
0	0	0
1	1	1
0	0	0
0	1	1

$\textit{The result of the bitwise operation is } 11010101$

$\textit{We are going to transform the result to decimal}$

$11010101 = 1 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$

$11010101 = 128 + 64 + 0 + 16 + 0 + 4 + 0 + 1$

$11010101 = 213$

$\textit{the result of the bitwise operation is } \mathbf{213}.$

(b) 217&95

$\frac{217}{2} = 108\ , r = 1$

$\frac{108}{2} = 54\ , r = 0$

$\frac{54}{2} = 27\ , r = 0$

$\frac{27}{2} = 13\ , r = 1$

$\frac{13}{2} = 6\ , r = 1$

$\frac{6}{2} = 3\ , r = 0$

$\frac{3}{2} = 1\ , r = 1$

$\frac{1}{2} = 0\ , r = 1$

$\frac{95}{2} = 47\ , r = 1$

$\frac{47}{2} = 23\ , r = 1$

$\frac{23}{2} = 11\ , r = 1$

$\frac{11}{2} = 5\ , r = 1$

$\frac{5}{2} = 2\ , r = 1$

$\frac{2}{2} = 1\ , r = 0$

$\frac{1}{2} = 0\ , r = 1$

$217\ in\ binary\ is\ 11011001\ and\ 95\ in\ binary\ is\ 1011111$

217	95	217&95
1	0	0
1	1	1
0	0	0
1	1	1
1	1	1
0	1	0
0	1	0
1	1	1

$The\ result\ of\ the\ bitwise\ operation\ is\ 1011001$

$1011001 = 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$ = **89.**

(c) 88^221

88^221

$\frac{88}{2} = 44\ , r = 0$

$\frac{44}{2} = 22\ , r = 0$

$\frac{22}{2} = 11\ , r = 0$

$\frac{11}{2} = 5\ , r = 1$

$\frac{5}{2} = 2\ , r = 1$

$\frac{2}{2} = 1\ , r = 0$

$\frac{1}{2} = 0\ , r = 1$

$\frac{221}{2} = 110\ , r = 1$

$\frac{110}{2} = 55\ , r = 0$

$\frac{55}{2} = 27\ , r = 1$

$\frac{27}{2} = 13\ , r = 1$

$\frac{13}{2} = 6\ , r = 1$

$\frac{6}{2} = 3\ , r = 0$

$\frac{3}{2} = 1\ , r = 1$

$$\frac{1}{2} = 0\,, r = 1$$

$$88\ in\ binary\ is\ 1011000\ and\ 221\ in\ binary\ is\ 11011101$$

88	221	88^221
0	1	1
1	1	0
0	0	0
1	1	0
1	1	0
0	1	1
0	0	0
0	1	1

$The\ result\ of\ the\ bitwise\ operation\ is\ 10000101$
$We\ are\ going\ to\ transform\ the\ result\ to\ decimal$
$10000101 = 1 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$
$10000101 = 128 + 0 + 0 + 0 + 0 + 4 + 0 + 1$
$10000101 = 133$
$The\ result\ of\ the\ bitwise\ operation\ is\ \mathbf{133}$

6) Explain how to utilize bitwise operators to solve the following problem: All students in a programming club are given a survey of 15 questions, each of the form, "Do you know language X?" Students must answer yes or no to each question. In scheduling lectures, the club would like to avoid lecturing on programming languages that every member in the club knows. For this problem, let's handle the slightly easier question of identifying the number of languages that are known by all members of the club. How can bitwise operators be used to

(a) Store the students' responses to the survey, and
(b) Calculate the answer to the given question?

Solution

The response of every student can be represented as a binary number of 15 bits where a 1 is a yes answer and a 0 is a no answer for the question “Do you know language X?”
The & operator can be used to calculate the number of languages that are known by all the students in the programming club. This can be done by creating an operation with all the survey questions with the operator being the & operator, so we would do
$Answer_1 \& Answer_2 \& Answer_3 \& \dots \dots . Answer_{n-1} \& Answer_n$
where there are n survey responses. The result of this bitwise operation would give a 15 bit binary number where the 1s represent the languages all students of the programming club know and where the 0s represent the languages not all students of the programming club know. We would just need to count the number of 1s to know the number of languages that all students in the programming club know.

7) The last question of each homework assignment will be to write up a two paragraph summary of a topic from the history of mathematics. The idea here is that rarely is any of this history taught in mathematics classes and while I don't have class time to teach it, I thought it would be nice if students learned a bit for each homework assignment. There's no need to use fancy sources, websites will do, but please site which websites you pulled your information from.

What are the Millennium Prize problems? Which is the only one of them to be deemed solved? Who is given credit for completing the solution? Give a summary of this person's life and contributions to mathematics.

Solution

The Millennium Prize problems are 7 mathematics problems selected by the Clay Mathematics Institute of Cambridge, which solutions are valued at 1 million dollars each. CMI announced the prizes in the year 2000 and they are some of the most difficult mathematics problems mathematicians have been struggling with. The prizes were announced to celebrate mathematics in the coming of the new century. The only Millennium Prize problem that has been solved thus far is the Poincaré conjecture, which was solved by Grigori Perelman.

Grigori Perelman is a Russian mathematician born the 13th of June of 1966 who has made contributions to Riemann geometry and geometric topology, but he is primarily known for proving the Poincaré conjecture. Furthermore, he proved the soul conjecture which is a theorem in Riemann geometry. Perelman was offered the Fields medal, which is one of the highest honors a mathematician can receive, and a million dollars, both for his role in proving the Poincaré conjecture. He rejected both offers, claiming that Richard S. Hamilton deserved as much credit as him. Hamilton came up with the process called Ricci flow, which was fundamental for Perelman's proof.

Sources:
Clay mathematics Institute, https://www.claymath.org/millennium-problems/millennium-prize-problems
Clay mathematics Institute, https://www.claymath.org/millennium-problems/poincar%C3%A9-conjecture
Wikipedia, https://en.wikipedia.org/wiki/Grigori_Perelman#Bibliography
Wikipedia, https://en.wikipedia.org/wiki/Ricci_flow
Wikipedia, https://en.wikipedia.org/wiki/Richard_S._Hamilton