Important Fall 2019 COT 3100 Final Exam Information

Exam Date and Time
Date: Thursday, December 5, 2019
Time: 10:00 am – 12:50 pm **Location: CB1-104**

NOTE THE EARLIER START TIME!!!

WHAT TO BRING
1. Two sheets of pre-prepared notes, either handwritten or typed (11" x 8.5"), both sides.

Exam Aids:

Two sheets of notes (regular 8.5”x11” paper, front and back either handwritten or typed), plus Exam #1 Formula Sheet. The latter will be provided for you. Please do NOT bring your own copy.

Exam Format
Time: 10 AM - 12:45 PM
All Free Response

COURSE TOPICS: GENERAL MATH - ALGEBRA, ETC. (BASED ON WARM-UPS), LOGIC, PROOF TECHNIQUES, SETS, NUMBER THEORY, SUMS, MATRICES, INDUCTION, COUNTING, PROBABILITY, RELATIONS, FUNCTIONS

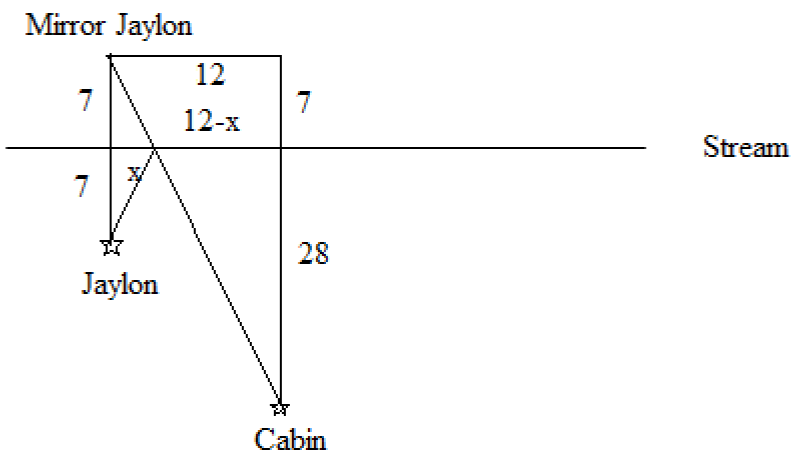
Questions and Solutions from 2018 Fall Exam

1) (8 pts) Jaylon is 7 miles south of a stream that flows due east. He is also 12 miles west and 21 miles north of his apartment. He wishes to go to the stream to collect some water and then return to his apartment. What's the minimum distance he can travel to accomplish this task? (Hint: the result is an integer number of miles.)

Solution

This problem is identical to the cowboy problem from the second to last recitation, with different substituted numbers. Here is the solution for convenience:

An equivalent problem would be if Jaylon was 7 miles ***north*** of the stream. Consider this picture:



Basically, since Jaylon has to get to the stream anyway, it doesn't really matter which side of the stream he comes from. Any path he could take to the stream from his original point, "Mirror Jaylon" can take an equivalent path of the same length and end at the exact same spot as our original cowboy. From there, their straight line paths to the cabin are identical.

It's fairly obvious that "Mirror Jaylon's" best path is the straight line that goes through the stream straight to the cabin which has length $\sqrt{12^2 + 35^2} = \mathbf{37}$ **miles.** Thus, our regular cowboy can simply aim to hit the stream at the same exact point and also obtain a 37 mile path. Any other path pursued by the regular cowboy will be longer. We can prove this by drawing the equivalent path of "Mirror Cowboy" which will end up being 2 sides of a triangle with 17 being the third side. (The triangle inequality shows that the sum of those two sides is greater than 17, and not the ideal path for the cowboy.)

Grading: For the solution path above: 7 pts to get to sqrt(1369), 1 pt to simplify to 37.

Any solution that picks some random point on the stream and calculates the correct traveling distance is worth 4 points. (1 or 2 pts off for incorrect calculation.)

The correct Manhattan distance (47 miles) gets 2 pts of credit. Setting up the correct Calculus equation to minimize gets 3 pts out of 8.

2) (8 pts) Find the ordered pair (x,y) that satisfy the pair of equations shown below:

$$log_2 x^3 + log_4 y^2 = 6$$

$$log_4 x^4 + log_2 y^6 = 20$$

Solution

Let A = log_2x and B = log_2y. Note the log base change rule, namely that $log_a b = \frac{log_c b}{log_c a}$. It follows that $log_4 x = \frac{log_2 x}{log_2 4} = \frac{log_2 x}{2} = \frac{A}{2}$. Similarly, $log_4 y = \frac{B}{2}$. Now take our given equations and apply the power rule to get:

$$3log_2 x + 2log_4 y = 6$$

$$4log_4 x + 6log_2 y = 20$$

Substituting so that our equations are in A and B now, we get:

$$3A + \frac{2B}{2} = 6$$

$$\frac{4A}{2} + 6B = 20$$

Now, solve the system:

3A + B = 6, so B = 6 - 3A

2A + 6B = 20

Substitute our expression for B from equation 1 into equation 2:

2A+6(6 - 3A) = 20
2A + 36 - 18A = 20
16A = 16
A = 1

It follows that B = 6 - 3(1) = 3.

To solve the problem, recall that A = log_2x, so x = 2^1 = 2 and B = log_2y, so y = 2^3. The desired solution is **(2, 8).**

Grading: Base change idea = 2 pts
Setting up system of equations = 1 pt
Solving system = 3 pts
Extracting final answer = 2 pts

3) (12 pts) Recall that $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$. We denote the n^{th} derivative of a function as $\frac{d^n y}{dx^n}$. Let $y = \sin(2x)$. Conjecture a guess for the function $\frac{d^{4n}y}{dx^{4n}}$, using the function y given, for all non-negative integers n. (Thus, your formula will be a formula for the 0^{th}, 4^{th}, 8^{th}, 12^{th}, etc. derivatives of $\sin(2x)$.) Prove your guess via mathematical induction on n.

Solution

The fourth derivative of sin(x) is sin(x). Using this understanding and the effect of the chain rule, we can surmise that for the given function, $\frac{d^{4n}y}{dx^{4n}} = 16^n \sin(2x)$.

Base case: n = 0, LHS = sin(2x), RHS = 16^0sin(2x) = sin(2x), this the base case holds.

Inductive hypothesis: Assume for an arbitrary non-negative integer n = k that

$$\frac{d^{4k}y}{dx^{4k}} = 16^k \sin(2x).$$

Inductive step: Prove for n = k + 1 that $\frac{d^{4(k+1)}y}{dx^{4(k+1)}} = 16^k \sin(2x)$.

$$\frac{d^{4(k+1)}y}{dx^{4(k+1)}} = \frac{d^4y}{dx^4}\left(\frac{d^{4k}y}{dx^{4k}}\right)$$

$$= \frac{d^4y}{dx^4}(16^k \sin(2x)), \text{ using the I. H.}$$

$$= \frac{d^3y}{dx^3}(16^k (2)\cos(2x))$$

$$= \frac{d^2y}{dx^2}(-16^k (2)(2)\sin(2x))$$

$$= \frac{dy}{dx}(-16^k (2)(2)(2)\cos(2x))$$

$$= (-1)(-1)16^k (2)(2)(2)(2)\sin(2x)$$

$$= 16^k (16)\sin(2x)$$

$$= 16^{k+1} \sin(2x)$$

Grading Criteria:

Guess = 4 pts (can give partial, 2 pts for sin(2x), 2 pts for 16^n or 2^{4n})
Base case = 1 pt (n=0 or n = 1 permitted)
IH = 1 pt
IS = 1 pt
Do 4 derivatives (2 pts) - note either order
Do IH (2 pts) - note either order
Simplify (1 pt)

4) (15 pts) A number of electronic coins have recently gained value. In particular, a bytecoin is worth 10 cents, a megacoin is worth one dollar, an opticoin is worth one dollar and finally a knightcoin is also worth one dollar. How many different combinations of bytecoins, megacoins, opticoins and knightcoins are worth exactly 20 dollars? Please leave your answer in powers, combinations, factorials, etc. and carefully explain what each expression in your answer and work represent.

Solution

Let x equal the number of bytecoins, y equal the number of megacoins, z equal the number of opticoins and w equal the number of knightcoins. Using the given info, we want the number of non-negative integer solutions to the equation:

$$10x + 100y + 100z + 100w = 2000$$

Divide this equation by 10:

$$x + 10y + 10z + 10w = 200$$

Since x = 200 - 10y - 10z - 10w, and y, z and w are integers, it follows that 10 | x. Create a new integer variable x' such that x = 10x', so our new equation reads, since x is divisible by 10:

$$10x' + 10y + 10z + 10w = 200$$

So now we want the number of non-negative integer solutions to the equation:

$$x' + y + z + w = 20$$

using the formula for combinations with repetition, the result is $\binom{20+4-1}{4-1} = \binom{23}{3}$.

An alternate way to solve the problem is to realize that since x can be anything in the equation x+ 10y + 10z + 10w = 200, that any solution to 10y + 10z + 10w ≤ 200, since we can treat x as our "overflow" variable for amount less than 200. Now, we want the number of solutions to 10y+ 10z + 10w ≤ 200. So, now just divide by 10 to get y + z + w ≤ 20. Finally, we learned that the way to solve this in class is to introduce a "slack" variable to account for the sum less than 20, so we are really looking for the total number of solutions to y + z + w + x' = 20, which is the same as above. (So, the result as well as the equation for which we are solving are the same, but this just shows an alternate line of reasoning to get there.)

Grading: 3 pts for setting up initial equation
2 pts for dividing by 10
4 pts for strategy to deal with x (either say mult 10 or slack var)
3 pts to reduce problem applying strategy
3 pts for applying combo with rep formula correctly

Valid summation = 10/15

5) (10 pts) What is the probability that a randomly chosen positive divisor of 15^{79} is an integer multiple of 45^{30}? Please express your answer as a fraction in lowest terms.

Solution

Note that $15^{79} = 3^{79}5^{79}$. It follows that the number of divisors this number has is $(79+1)(79+1) = 80^2$. (Recall that all divisors are of the form 3^a5^b with $0 \le a,b \le 79$. So the number of divisors is equal to the number of ordered pairs (a, b) that satisfy the constraints given. Since the choices of a and b are independent, we simply multiply the number of possible choices of by the number of possible choices by b. For both, the possible number of choices equals the number of integers in between 0 and 79, inclusive, which is 80.)

Thus, the sample space for our problem is 80^2, since there are 80^2 divisors of 15^{79}.

Now, let's prime factorize $45^{30} = (3^2)^{30}5^{30} = 3^{60}5^{30}$. In order for one of the divisors of 15^{79} to be a multiple of 45^{30}, we require it to be of the form 3^a5^b where $a \ge 60$ and $b \ge 30$. So, we must count the number of divisors of 15^{79} of the form 3^a5^b where $a \ge 60$ and $b \ge 30$. Recall that we also have $a \le 79$ and $b \le 79$. Thus, we simply desire the number of ordered pairs (a, b) with $60 \le a \le 79$, and $30 \le b \le 79$. Again, the choices of a and b are independent, so we can count the number of ordered pairs by simply multiplying the number of integers a that satisfy the given inequality by the number of integers b that satisfy its given inequality. There are $(79 - 60 + 1) = 20$ possible values of a and $(79 - 30 + 1) = 50$ possible values of b, for a total of 20 x 50 possible divisors of 15^{79} that are also multiples of 45^{30}.

It follows that our desired probability is $\frac{20\times50}{80\times80} = \frac{10}{64} = \mathbf{\frac{5}{32}}$.

Grading: 4 pts for sample space (can give partial here if progress is made).
4 pts for figuring out which items in the sample space are multiples of 45^{30}.
1 pt for writing as a fraction (with smaller # over bigger #)
1 pt for reducing to lowest terms

6) (10 pts) Let $f(x) = \frac{3x-2}{x+5}$, with a domain of all reals except x = -5. Determine $f^{-1}(x)$ as well as the domain and range of $f^{-1}(x)$.

Solution

Switch x and y and solve for y:

$$x = \frac{3y-2}{y+5}$$
$$x(y+5) = 3y-2$$
$$xy + 5x = 3y - 2$$
$$5x + 2 = 3y - xy$$
$$5x + 2 = y(3-x)$$
$$y = \frac{5x+2}{3-x}$$

It follows that $\boldsymbol{f^{-1}(x) = \frac{5x+2}{3-x}}$. The domain of this inverse function is the range of the original function, it's **all reals except for x = 3.** The range of this function is the domain of the original function which is **all reals except for y = -5.**

Grading: 2 pts to switch x and y
6 pts to solve for y (give partial as you see fit)
1 pt for domain (has to be 100% correct to get the point)
1 pt for range (has to be 100% correct to get the point)

7) (12 pts) Define a relation R over the set of rational numbers as follows: R = { (p, q) | p - q has a denominator with absolute value less than 200 when reduced to lowest terms }. Determine, with proof, whether or not R is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric and (e) transitive.

Solution

The relation is reflexive. For any rational number p, $p - p = \frac{0}{1}$, thus, the corresponding denominator in question is less than 200 and (p, p) ∈ R, for all rational numbers p.

The relation is NOT irreflexive because $(\frac{1}{1}, \frac{1}{1})$ belongs to the relation (as an example of the general case previously stated.)

The relation is symmetric because if (p, q) is in the relation, then $p - q = \frac{m}{n}$, where |n| < 200. We can multiply this equation through by -1 to yield $q - p = \frac{-m}{n}$. Since m is an integer, -m is also an integer and $\frac{-m}{n}$ is a rational number. Finally, since |n| < 200, this result is a rational number with a denominator less than 200. From this equation, it follows that (q, p) ∈ R, as desired to prove that R is symmetric.

R is NOT anti-symmetric. Note that $(\frac{2}{1}, \frac{1}{1}) \in R$ and $(\frac{1}{1}, \frac{2}{1}) \in R$, but that $\frac{2}{1} \neq \frac{1}{1}$, and that the difference between these fractions is $\frac{1}{1}$ and $\frac{-1}{1}$, respectively, both with denominators less than 200.

R is NOT transitive. Note that $(\frac{1}{21}, \frac{1}{35}) \in R$ and $(\frac{1}{35}, \frac{1}{20}) \in R$, but $(\frac{1}{21}, \frac{1}{20}) \notin R$. Notice that gcd(21, 35) = 7, so lcm(21, 35) = 105 and gcd(35, 20) = 5, so lcm(35, 20) = 140, but since gcd(21, 20) = 1, lcm(21, 20) = 420. (The denominator of the difference will generally equal the lcm of the original denominators.)

Grading: 2 pts for reflexive, irrflexive, symmetric, anti-symmetric, 4 pts for transitive. (1 pt for getting whether or not it is that quality right, 1 pt for proof of the first four, 3 pts for proof of the last one)