

Fall 2019 COT 3100 Exam #2 10/24/2019 Solutions

1) (10 pts) Eileen rides her bike a distance of D miles at a rate of r miles per hour from home to the bus stop. She then takes the bus for a distance of $5D$ miles traveling at a rate of $4r$ miles per hour to arrive at work. Assume that the bus starts immediately after Eileen arrives to the bus stop. If the average speed for Eileen's trip from home to work is 32 miles per hour, at what rate, in miles per hour, does she ride her bike for that portion of the trip?

Thus, the goal is to find the value of r defined in the question.

The total time taken on the trip is $\frac{D}{r} + \frac{5D}{4r} = \frac{9D}{4r}$ hours.

The total distance for the trip is simply $D + 5D$ miles.

Thus, we can express the average speed of the trip as $\frac{6D}{9D/4r} = \frac{24}{9}r = \frac{8r}{3}$ miles per hour.

In the question we are given that this is equal to 32 miles per hour. Equating, we can solve for r :

$$\frac{8r}{3} = 32$$

$$r = \frac{3 \times 32}{8} = 12 \text{ miles per hour}$$

Grading: 3 pts for expression of total time on trip, 2 pts for total distance, 3 pts for average speed of whole trip in terms of r , 2 pts solving for r .

2) (12 pts) Consider a particle on the Cartesian plane that starts at point $(0, 0)$ at time $t=0$, initially headed in the direction of the positive y-axis. At time t , the particle will move $t+1$ units in the direction that it is headed, with the first movement starting at $t=0$. Thus, at $t=1$, the particle will end up at $(0,1)$. After each move, the particle will turn left 90 degrees, instantaneously. Thus, at $t=2$, the particle will end up at $(-2, 1)$, since it's headed in the direction of the negative x-axis and travels 2 units in that direction. Using induction on n , prove for all non-negative integers n , that at time $t = 4n$, the particle will be located at $(2n, -2n)$, heading in the positive y-axis.

Using induction on n , we will show that at time $t = 4n$, the particle will be located at $(2n, -2n)$, for all non-negative integers n .

Base case: $n = 0$. At time $n = 0$, the particle is at $(0, 0)$ as defined in the problem. Plugging in $n = 0$ to the given point we get $(2(0), -2(0))$ which is also $(0, 0)$. Thus the base case holds.

Inductive Hypothesis: Assume for an arbitrarily chosen non-negative integer $n = k$ that at time $4k$, the particle will be located at $(2k, -2k)$ heading in the positive y-axis.

Inductive Step: Prove, for $n = k+1$, that at time $4(k+1) = 4k+4$, the particle will be located at $(2(k+1), -2(k+1))$, heading in the positive y-axis.

Using the IH, we know that at time $4k$, the particle is at $(2k, -2k)$ heading in the positive y-axis. At time $4k+1$, the particle will be at $(2k, -2k+4k+1)$ which is $(2k, 2k+1)$, since it just went "up". Next, at time $4k+2$, the particle will be at $(2k - (4k+2), 2k+1)$, which is $(-2k-2, 2k+1)$, since after going up, the particle goes left (in the direction of the negative x-axis). At time $4k+3$, the particle will be at $(-2k-2, 2k+1-(4k+3))$ which is $(-2k-2, -2k-2)$. Finally, at time $4k+4$, the particle heads $4k+4$ units in the direction of the positive x-axis arriving at the point $(-2k-2+4k+4, -2k-2)$, which simplifies to $(2k+2, -2k-2)$. This simplifies to $(2(k+1), -2(k+1))$, proving the inductive step.

Note: An easier explanation is that in the first and third move after time $t=4k$, the particle will move $4k+1$ in the positive y-axis followed by $4k+3$ in the negative y-axis for a net movement of -2 in y . Similarly, the particle will move $4k+2$ in the negative x-axis followed by $4k+4$ in the positive x-axis 2 time units later for a net movement of $+2$ in x . Thus, after 4 moves from $(2k, -2k)$, the particle will end up at $(2k + 2, -2k - 2)$ which is $(2(k+1), -2(k+1))$.

Grading: 2 pts for the base case, 2 pts for the IH, 2 pts for the IS, 2 pts for invoking the IH and starting the simulation at time $t = 4k$, 3 pts for calculating where the particle will be at time $4k+4$, 1 pt for converting this into an equivalent form of the IH or stating that it's equivalent.

3) (15 pts) Find all integer solutions to the equation $258x + 204y = 42$.

Run the Euclidean Algorithm with $a = 258$ and $b = 204$:

$$\begin{array}{ll} 258 = 1 \times 204 + 54 & 43 = 1 \times 34 + 9 \\ 204 = 3 \times 54 + 42 & 34 = 3 \times 9 + 7 \\ 54 = 1 \times 42 + 12 & 9 = 1 \times 7 + 2 \\ 42 = 3 \times 12 + 6 & 7 = 3 \times 2 + 1 \\ 12 = 2 \times 6 & \end{array}$$

Since $6 \mid 42$, divide the given equation through by 6 to get the equivalent equation:

$$43x + 34y = 7$$

Run the Extended Euclidean with 43 and 34. Above the Euclidean is written with each relevant number divided by 6 so that the equations below make sense:

$$\begin{array}{l} 7 - 3 \times 2 = 1 \\ 7 - 3(9 - 1 \times 7) = 1 \\ 7 - 3 \times 9 + 3 \times 7 = 1 \\ 4 \times 7 - 3 \times 9 = 1 \\ 4(34 - 3 \times 9) - 3 \times 9 = 1 \\ 4 \times 34 - 12 \times 9 - 3 \times 9 = 1 \\ 4 \times 34 - 15 \times 9 = 1 \\ 4 \times 34 - 15(43 - 34) = 1 \\ 4 \times 34 - 15 \times 43 + 15 \times 34 = 1 \\ 19 \times 34 - 15 \times 43 = 1 \end{array}$$

Now, multiply this equation through by 7:

$$(19 \times 7) \times 34 - (15 \times 7) \times 43 = 7$$

Thus, one solution is $(-105, 133)$. If (x, y) is one solution to the equation above, then for any integer c , $(x + 34c, y - 43c)$ is also another solution. It follows that the full solution set is:

$$\{ (x, y) \mid (-105 + 34c, 133 - 43c), \forall c \in \mathbb{Z} \}$$

Note that we can plug in any value for c and use this as our canonical solution. A “popular” way to represent this solution may be to plug in $c = 3$ which yields $\{(x, y) \mid (-3 + 34c, 4 - 43c), \forall c \in \mathbb{Z}\}$

Grading: 4 pts reg Euclidean, 1 pt divide through by 6 or equivalent (you can work with the original equations also...), 7 points Extended Euclidean, 1 pt mult by 7, 2 pts to use this to get to final answer

4) (10 pts) Let x and y be integers such that $19 \mid (5x - y)$. Prove that $19 \mid (x+15y)$.

$$\begin{aligned}x + 15y &= 20x - 19x - 4y + 19y \\&= 20x - 4y - 19(x - y) \\&= 4(5x - y) - 19(x - y), \text{ using the given information, there exists an integer } c \text{ such that} \\&\quad 5x - y = 19c, \text{ so substitute.} \\&= 4(19c) - 19(x - y) \\&= 19(4c - x + y),\end{aligned}$$

Since c , x and y are all integers, so is $4c - x + y$. It follows that $19 \mid (x + 5y)$, as desired.

Grading: 7 pts to get the breakdown shown above (or any suitable breakdown), 2 pts to sub in $19c$ for $5x - y$, 1 pt to finish off the proof.

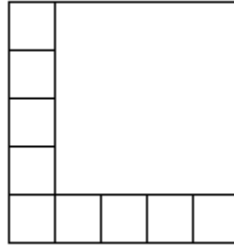
Note: In class I showed a systematic way to solve for the breakdown in line 1:

There is some integer d such that

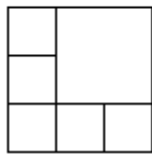
$$\begin{aligned}d(5x - y) &\equiv x + 15y \pmod{19} \\5dx - dy &\equiv x + 15y \pmod{19}\end{aligned}$$

Equating coefficients, we have $5d \equiv 1 \pmod{19}$ and $-d \equiv 15 \pmod{19}$. Solving the latter is simple, just multiply through by -1 to yield $d \equiv -15 \equiv 4 \pmod{19}$. This indicates that in order to express $x + 15y$, we should take $4(5x - y)$ and then adjust by multiples of 19 of x and y , separately. If a student attempts this, it counts as part of the 7 points to achieve the breakdown. Even if they fail to do so, in trying this method students can earn partial credit of these 7 points.

5) (15 pts) Using strong induction on n with three base cases, prove that a square can be partitioned into n squares, for all positive integers $n \geq 6$. (Note: To partition a square, you must draw some line segments dividing the square so that all of the separate pieces are non-intersecting, cover the whole square, and are all squares themselves. The picture below shows a square partitioned into 10 squares. Specifically in this picture, if the original square has side length s , 9 of the squares have side length $\frac{s}{5}$, and the tenth square in the upper right hand corner has side length $\frac{4s}{5}$.) Your proof should primarily have pictures accompanied with the specific dimensions of each of the smaller squares in terms of the square to be partitioned.

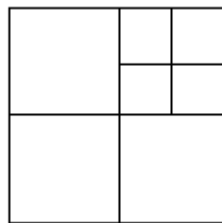


The base cases for $n = 6$, $n = 7$ and $n = 8$ are drawn below. Given that the original square has side length s , the text below specifies the side lengths of each of the smaller squares that comprise the original larger square.



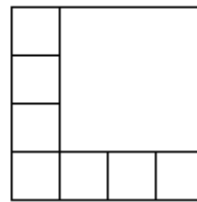
$n = 6$

5 sq w/side len $s/3$
1 sq w/side len $2s/3$



$n = 7$

3 sq w/side len $s/2$
4 sq w/side len $s/4$



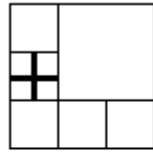
$n = 8$

7 sq w/side len $s/4$
1 sq w/side len $3s/4$

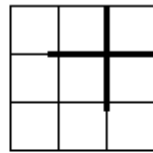
Inductive hypothesis: for all integers n , such that $6 \leq n \leq k$, where k is an arbitrarily chosen integer greater than or equal to 8, there exists a way to partition a square into n smaller squares.

Inductive step: Prove, for $n = k+1$, that there exists a way to partition a square into $k+1$ smaller squares.

Since $k \geq 8$, it follows that $k+1 \geq 9$. It follows from this inequality that $k - 2 \geq 6$. Thus, the inductive hypothesis can be applied to $k-2$, meaning that there exists some way to partition a square into $k-2$ smaller squares. Take this particular partition, and arbitrarily pick any of the smaller squares in it (it doesn't matter which one...). Then, partition this one smaller square into 4 squares by drawing in both the vertical and horizontal lines of symmetry for the square (the natural way to split a square into 4 squares...). For example, if $k+1 = 9$, then $k - 2 = 6$, and we can take the leftmost picture shown in the base cases and then subdivide any of those squares into four squares. Here are two such possibilities. The new square division is highlighted with wider line segments:



Divide $s/3$ sq
into 4 sqs.



Divide $2s/3$ sq
into 4 sqs.

Both pics have 9 squares

Grading criteria: 3 pts $n = 6$, 3 pts $n = 7$, 3 pts $n = 8$ (2 pts for a base case that is mislabeled but correct), 1 pt IH, 1 pt IS (to get IH point, must state for all $n \leq k$ and k arbitrarily chosen), 4 pts for inductive proof (NO POINTS FOR SPLITTING INTO ODD AND EVEN CASES as this doesn't work for the induction...)

6) (14 pts) Let p be an positive integer (greater than 1) such that $2^p - 1$ is a prime number. Let $n = 2^{p-1}(2^p - 1)$. Determine the sum of divisors of n , in terms of either n , p or both. (**Note: Full credit is given only for the most simple form of the final answer.**)

Using the formula given in class, the sum of divisors of n can be broken down as follows:

$$\frac{2^p - 1}{2 - 1} \times (1 + 2^p - 1) = (2^p - 1)(2^p) = (2^p - 1)(2^{p-1})(2) = n(2) = 2n$$

Note that in the second expression before the first equal sign, instead of plugging into the usual formula which would give us $\frac{(2^p - 1)^2 - 1}{2^p - 1 - 1}$, I decided to plug directly into the original summation which adds up 1 plus the prime in question, plus the prime to each power upto the exponent in the prime factorization. But, since this exponent is just one, the summation form is just two simple terms. Whereas the expression above is clearly more complicated. Eventually after squaring and simplifying, we'll get to 2^p .

Note: a perfect number is a number whose sum of **proper divisors** is equal to the number itself. Notice that if we subtract the number, n itself, out of the sum of all of its divisors $2n$, we get n , the number itself. Thus, all numbers of this form (where $2^p - 1$ is prime) are perfect numbers, as proven by the response to this question. It turns out that all even perfect numbers are of this form and that no one has discovered any odd perfect numbers yet, but if one exists, it must be "pretty big." Also, it's unknown if there are an infinite number of primes of the form $2^p - 1$. If there are, then there are also an infinite number of perfect numbers. Primes of this form are called Mersenne Primes.

Grading: 5 pts for the part of the product for 2^{p-1} , 4 pts for the second part of the product for $2^p - 1$, 3 pts to simplify the terms to the form $(2^p - 1)$ and 2^p , respectively, 2 pts to arrive at the final form, recognizing that n is hiding in the expression =)

7) (12 pts) Using induction on n , prove for all non-negative integers n that $\sum_{i=0}^n \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

Base case: $n = 0$, LHS = $\sum_{i=0}^0 \frac{i}{2^i} = \frac{0}{2^0} = 0$, RHS = $2 - \frac{0+2}{2^0} = 2 - \frac{2}{1} = 2 - 2 = 0$

Thus, the base case holds.

Inductive hypothesis: Assume for an arbitrarily chosen non-negative integer $n = k$ that

$$\sum_{i=0}^k \frac{i}{2^i} = 2 - \frac{k+2}{2^k}$$

Inductive step: Prove for $n = k+1$ that

$$\sum_{i=0}^{k+1} \frac{i}{2^i} = 2 - \frac{(k+1)+2}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}}$$

$\sum_{i=0}^{k+1} \frac{i}{2^i} = \sum_{i=0}^k \frac{i}{2^i} + \frac{(k+1)}{2^{k+1}}$, splitting the sum

$$= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}, \text{ using the inductive hypothesis}$$

$$= 2 - \frac{2(k+2)}{2(2^k)} + \frac{k+1}{2^{k+1}}, \text{ getting a common denominator}$$

$$= 2 - \frac{[2k+4-(k+1)]}{2^{k+1}}$$

$$= 2 - \frac{[2k+4-k-1]}{2^{k+1}}$$

$$= 2 - \frac{k+3}{2^{k+1}}, \text{ completing the proof of the inductive step.}$$

It follows that for all non-negative integers n , $\sum_{i=0}^n \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

Grading: 2 pts BC, 2 pts IH, 2 pts IS, 1 pt split sum, 2 pts plug in IH, 3 pts rest of the algebra

8) (10 pts) On Bobby's first day of work, he earns \$100. On every subsequent day, he earns \$4 more than the previous day. On Selena's first day of work, she earns \$10. On every subsequent day, she earns \$6 more than the previous day. Both start work on the same day. After n days of both working, they compare their total earnings to that point in time and realize that Selena has earned strictly more money in total than Bobby. What is the minimum possible value of n ?

On the n^{th} day Bobby will make $100 + 4(n-1)$ dollars and Selena will make $10 + 6(n-1)$ dollars.

Over the course of n days, Bobby's average pay per day will be the sum of his first and last day's worth of pay divided by 2, since his pay is an arithmetic sequence, thus, his average pay per day is just $\frac{100+100+4(n-1)}{2} = 100 + 2(n-1) = 100 + 2n - 2 = 2n + 98$.

Selena's average pay per day is $\frac{10+10+6(n-1)}{2} = 10 + 3(n-1) = 10 + 3n - 3 = 3n + 7$.

Since the two work the same number of days, we can solve the problem by finding the minimum value of n such that Selena's average pay per day is strictly greater than Bobby's:

$$3n + 7 > 2n + 98$$

$$n > 91$$

Thus, the minimum value of n that satisfies the requirement is $n = 92$, since n must be an integer.

Grading: 2 pts for how much Bobby gets paid on day n , 2 pts for how much Selena gets paid on day n , 2 pts for either calculating Bobby's average pay per day or total pay after n days, 2 pts for calculating the same for Selena, 2 pts for solving the inequality and answering with the minimum value of n .

Note: Most students will opt to find the total pay of both Bobby and Selena instead of the average pay.

9) (2 pts) By what acronym is the National Basketball Association typically referred? **NBA, give to all.**