

Fall 2019 COT 3100 Exam #1 (9/27/2019) (Note: Out of 75 points) Solutions

1) (10 pts) Use the laws of logic to show that the two following expressions are logically equivalent:

$$(a) (q \rightarrow p) \vee (r \wedge (\bar{r} \wedge (\overline{p \vee r})))$$

$$(b) p \vee \bar{q}$$

Note: You may not use all of the rows shown below.

Step	Reason
1. $(q \rightarrow p) \vee (r \wedge (\bar{r} \wedge (\overline{p \vee r})))$	Given
2. $(q \rightarrow p) \vee (r \wedge (\bar{r} \wedge (\bar{p} \wedge \bar{r})))$	De Morgan's Law
3. $(q \rightarrow p) \vee (r \wedge ((\bar{r} \wedge \bar{r}) \wedge \bar{p}))$	Associative Law
4. $(q \rightarrow p) \vee (r \wedge (\bar{r} \wedge \bar{p}))$	Idempotent Law
5. $(q \rightarrow p) \vee ((r \wedge \bar{r}) \wedge \bar{p})$	Associative Law
6. $(q \rightarrow p) \vee (F \wedge \bar{p})$	Inverse Law
7. $(q \rightarrow p) \vee F$	Domination Law
8. $q \rightarrow p$	Identity Law
9. $\bar{q} \vee p$	Implication Identity
10. $p \vee \bar{q}$	Commutative Law

Grading: If the response is generally complete, take off 1 pt per error.

If the response is fairly far away from being correct, give 1 pt per valid step.

If reasons are incorrect, take off 1/2 pt, round down. Cap at 3 pts off total.

2) (10 pts) Use the rules of inference (and potentially the laws of logic) to make the following argument:

$$s \rightarrow (r \vee \bar{p})$$

$$\bar{q}$$

$$t \rightarrow u$$

$$\bar{p} \rightarrow q$$

$$u \rightarrow s$$

$$t$$

$$r$$

Step	Reason
1. t	Premise
2. $t \rightarrow u$	Premise
3. u	Modus Ponens with 1, 2
4. $u \rightarrow s$	Premise
5. s	Modus Ponens with 3, 4
6. $s \rightarrow (r \vee \bar{p})$	Premise
7. $r \vee \bar{p}$	Modus Ponens with 5, 6
8. $\bar{p} \rightarrow q$	Premise
9. $\bar{q} \rightarrow \bar{\bar{p}}$	Contrapositive of 8
10. $\bar{q} \rightarrow p$	Double Negation on 9
11. \bar{q}	Premise
12. p	Modus Ponens
13. r	Disjunctive Syllogism with 7, 12

Grading: 3 pts for all the premises, 1 pt each for each of the other steps, take off partial similar to question #1. Max 3 pts off for missing reasons.

3) (10 pts) Let n be a positive integer such that $10 \mid (n - 9)$. Prove that $8 \mid (n^2 - 1)$. In your proof, you may use the result that for all integers a , $a(a+1)$ is an even integer.

Solution

By definition of divisibility, there exists an integer a such that

$$n - 9 = 10a. \text{ Thus,}$$

$$n = 10a + 9, \text{ for some integer } a.$$

$$\begin{aligned} n^2 - 1 &= (10a + 9)^2 - 1 \\ &= 100a^2 + 180a + 81 - 1 \\ &= 96a^2 + 4a^2 + 176a + 4a + 80 \\ &= 96a^2 + 176a + 80 + 4a^2 + 4a \\ &= 8(12a^2 + 22a + 10) + 4a(a+1) \end{aligned}$$

As previously proved in class, for any integer a , $a(a+1)$ is even. Thus, there exists some integer b such that $a(a+1) = 2b$. Substituting, we get:

$$\begin{aligned} &= 8(12a^2 + 22a + 10) + 4(2b) \\ &= 8(12a^2 + 22a + 10 + b) \end{aligned}$$

Since a and b are integers, it follows that $(12a^2 + 22a + 10 + b)$ is an integer. We can conclude then, that the original quantity, $n^2 - 1$, is divisible by 8.

Grading: 3 pts to get to $n = 10a + 9$, for some integer a .

1 pt for plugging that into $n^2 - 1$,

2 pts for properly foiling it out

2 pts for factoring 8 out of some subset of stuff

2 pts for getting the rest to have 8 factored out

4) (10 pts) A school offers three clubs: Art Club, Debate Club and Chess Club. There are 22 students who are either in Art Club or Debate Club and there are 25 students who are either in Debate Club or Chess Club. However, there are no students in both Art and Chess Club. If there are 16 students who are in Debate Club, how many students are in at least one of the three clubs? In order to receive full credit, you must utilize the Inclusion-Exclusion Principle for two and three sets formally. You may draw Venn Diagrams to assist you in coming up with your solution and such a diagram may earn a small amount of partial credit, but most of the credit is earmarked for formally using the Inclusion-Exclusion Principle.

Solution

Let the set A denote the Art Club, the set B denote the Debate Club and the set C denote the Chess Club. Converting each of the given facts into mathematical statements yields the following:

$$\begin{aligned} |A \cup B| &= 22 \\ |B \cup C| &= 25 \\ |A \cap C| &= 0 \\ |B| &= 16 \end{aligned}$$

Our goal is to establish the value of $|A \cup B \cup C|$. Take the first two statements and expand them using the Inclusion-Exclusion Principle:

$$\begin{aligned} 22 &= |A \cup B| = |A| + |B| - |A \cap B| \\ 25 &= |B \cup C| = |B| + |C| - |B \cap C| \end{aligned}$$

Adding these two equations, we get:

$$47 = |A| + 2|B| + |C| - |A \cap B| - |B \cap C|$$

Substitute $|B| = 16$, for one copy of $|B|$:

$$\begin{aligned} 47 &= |A| + |B| + 16 + |C| - |A \cap B| - |B \cap C| \\ 31 &= |A| + |B| + |C| - |A \cap B| - |B \cap C| \end{aligned}$$

Now, let's substitute into the Inclusion-Exclusion Principle for 3 sets to solve the given question:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| - |A \cap B \cap C|$$

Now, note that if $|A \cap C| = 0$, then $|A \cap B \cap C| = 0$, since $(A \cap B \cap C) \subseteq (A \cap C)$. Thus, we can plug in for the first 5 terms using the previous deduction and the last two terms are 0. Thus.

$$|A \cup B \cup C| = 31 - 0 - 0 = \mathbf{31}$$

Grading: 1 pt for writing down I/E for sets A,B, 1 pt for writing down I/E for sets B,C, 3 pts for adding equations, 2 pts for subbing $|B| = 16$, 2 pts for realizing that $A \cap B \cap C$ is empty, and 1 pt for plugging into the big I/E to get the answer. (3 pts for correct answer with Venn Diagram only.)

5) (8 pts) Prove or Disprove: For arbitrary sets, A, B, C and D,

$$\text{if } (A \cup B) \subseteq (C \cap D), \text{ then } A \subseteq C \wedge B \subseteq D.$$

Solution

This statement is true. We have two things to prove:

(1) $A \subseteq C$

(2) $B \subseteq D$

Let's tackle each separately.

To prove the first, we start with an arbitrarily chosen element x such that $x \in A$. We aim to prove that $x \in C$.

1. Note that if $x \in A$, by definition of union $x \in A \cup B$.
2. Since $(A \cup B) \subseteq (C \cap D)$, by definition of subset we have $x \in C \cap D$.
3. Finally, by definition of intersection, we conclude that $x \in C$.

This completes our proof of (1).

To prove the second, we start with an arbitrarily chosen element x such that $x \in B$. We aim to prove that $x \in D$.

1. Note that if $x \in B$, by definition of union $x \in A \cup B$.
2. Since $(A \cup B) \subseteq (C \cap D)$, by definition of subset we have $x \in C \cap D$.
3. Finally, by definition of intersection, we conclude that $x \in D$.

This completes our proof of (2). Thus, the entire given statement is proven true.

Grading: 5 pts for proving one of the two things (either $A \subseteq C$ or $B \subseteq D$), 3 pts for the second one.

For first proof: 1 pt for picking an arbitrary element in A or B.

1 pt to argue is in A union B

2 pts to argue it's in C intersect D

1 pt to argue it's in in C.

For second proof, if it's there it'll probably be full credit, if you need to give partial out of 3, do so.

6) (8 pts) Prove or Disprove: For arbitrary sets, A, B, C,

$$\text{if } A \cap B \cap C = \emptyset, \text{ then } (A \subseteq \bar{B}) \vee (A \subseteq \bar{C})$$

Solution

This is false. Consider the following counter-example:

$$A = \{1,2\}, B = \{1\}, C = \{2\}$$

For this example, we see that $A \cap B \cap C = \emptyset$, since no one element is contained in all three sets.

But, it's also the case that $A \subseteq \bar{B}$ is false because 1 is an element of A but NOT an element of \bar{B} . (Everything that is in B is NOT in \bar{B} and 1 is an element of B.)

In addition, $A \subseteq \bar{C}$ is false because 2 is an element of A but NOT an element of \bar{C} .

Grading: Max 3 pts for any proof.

4 pts for just saying it's false.

3 pts for a valid counter-example

1 pt for explaining why it's a valid counter-example.

7) (10 pts) Determine the ordered pair, (a, b), that satisfies the following pair of equations:

$$\log_{16}a^2 + \log_8b^3 = 11$$

$$\log_8a^6 + \log_{16}b^{10} = 32$$

Note: You may express a and b as a some base raised to an exponent instead of a single value, if you wish.

Solution

Take a common base of 2 for all the logs to yield the following:

$$\frac{\log_2a^2}{\log_216} + \frac{\log_2b^3}{\log_28} = 11$$

$$\frac{\log_2a^6}{\log_28} + \frac{\log_2b^{10}}{\log_216} = 32$$

Simplify using the power rule and evaluating constants:

$$\frac{2}{4}\log_2a + \frac{3}{3}\log_2b = 11$$

$$\frac{6}{3}\log_2a + \frac{10}{4}\log_2b = 32$$

Let $x = \log_2a$, $y = \log_2b$. Substituting (and multiplying the first equation through by 4 we get:

$$2x + 4y = 44$$

$$2x + \frac{5y}{2} = 32$$

Subtracting the bottom from the top yields $\frac{3y}{2} = 12$, so $y = 8$. Plugging this back into the first equation yields $2x + 32 = 44$, so $x = 6$. Back substituting to solve for a and b we get:

$a = 2^6 = 64$, $b = 2^8 = 256$. The desired ordered pair is **(64, 256)**.

Grading: 2 pts to change log correctly to any base, 3 pts to change it to base 2.

2 pts to use power rule and get to equations in log a and log b.

5 pts to solve the system of equations

1 pt to substitute to get the correct answers for a and b themselves.

8) (6 pts) Jessica is going from Orlando to Miami and she takes a 15 minute break at Vero Beach. Her goal is to average 60 miles per hour for the whole trip. The distance between Orlando and Vero Beach is 100 miles and the distance between Vero Beach and Miami is 140 miles. If her average driving speed from Orlando to Vero Beach is 50 miles per hour, how fast must her average speed be driving from Vero Beach to Miami to achieve her goal? (**Note: I don't actually advise driving this fast ever!!!**)

Solution

The total distance to drive is 240 miles and if we aim to achieve an average of 60 miles per hour, we must make the drive in $240 \text{ miles}/60 \text{ mph} = 4 \text{ hours}$.

If we drive 100 miles averaging 50 miles per hour, we take $100 \text{ miles}/50 \text{ mph} = 2 \text{ hours}$.

Then we take a break for 15 minutes. This means we've used 2 hours and 15 minutes.

We have $4 \text{ hours} - (2 \text{ hours } 15 \text{ minutes}) = 1 \text{ hour } 45 \text{ minutes} = \frac{7}{4} \text{ hours}$ to complete the trip.

Let r be the average speed we drive from Vero Beach to Orlando. Then we have:

$$140 \text{ miles} = r\left(\frac{7}{4} \text{ hours}\right)$$
$$r = 140 \times \frac{4}{7} \text{ mph} = \mathbf{80 \text{ mph}}$$

Grading: 2 pts to figure out whole trip should take 4 hours.

2 pts to figure out that we have 1 3/4 hours to complete Vero to Miami.

2 pts to calculate the correct answer from this point.

9) (3 pts) What holiday does Universal Studio's Halloween Horror Nights celebrate?

Halloween

Grading: Give to all 3 pts