COT 3100 Fall 2018 Homework 10 Solutions
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1) Let R_1 and R_2 be relations on a set A = {1, 2, 3, 4}.
In particular, let R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3) (4, 4)} and R_2 = {(1, 2), (2, 3), (3, 4), (1, 3), (2, 4)}.

Determine the following:

a) Whether or not R_1 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
b) Whether or not R_2 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
c) The relation $R_1 \circ R_2$.
d) The relation $R_2 \circ R_1$.
e) $R_1 \cup R_2$
f) $R_1 \cap R_2$
g) The reflexive, symmetric and transitive closures of both R_1 and R_2.

Solution

a) R_1 is reflexive, because it contains all ordered pairs of the form (a, a) for all a in A.
R_1 is NOT irreflexive because it contains (1, 1).
R_1 is symmetric, for each ordered pair of the form (a, b), it also contains (b, a).
R_1 is not anti-symmetric since it contains both (1, 2) and (2, 1) and $1 \neq 2$.
R_1 is transitive, because for all ordered pairs (a, b) and (b, c) in R_1, (a, c) is also in R_1.
(To see this, note that if a = 1 or 2, b = 1 or 2 and if b = 1 or 2, c = 1 or 2. Similarly, if a = 3 or 4, then b = 3 or 4 and c = 3 or 4.)

b) R_2 is not reflexive because (1,1) is not in R_2.
R_2 is irreflexive because for all a in A, it doesn't contain (a, a).
R_2 is not symmetric because it contains (1, 2) but not (2, 1).
R_2 is anti-symmetric, for all (a, b) in R_2 with $a \neq b$, (b, a) is NOT in R_2.
R_2 is NOT transitive. It contains (1, 3) and (3, 4) but not (1, 4).

c) $R_1 \circ R_2$ = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)}.

d) $R_2 \circ R_1$ = {(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 4)}.

e) $R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(2,4),(3,3),(3,4),(4,3)(4,4)\}$

f) $R_1 \cap R_2 = \{(1,2),(3,4)\}$

g) $r(R_1) = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3)(4,4)\}$
$s(R_1) = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3)(4,4)\}$
$t(R_1) = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3)(4,4)\}$
$r(R_2) = \{(1,1),(1,2),(2,2),(2,3),(3,3),(3,4),(4,4),(1,3),(2,4)\}$
$s(R_2) = \{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),(1,3),(3,1),(2,4),(4,2)\}$
$t(R_2) = \{(1,2),(2,3),(3,4),(1,3),(2,4),(1,4)\}$

2) Let $A \subseteq Z$ be a subset of integers and let R be a relation over $A \times A$ defined as follows:

$$(x_1, x_2) R\ (y_1, y_2) \text{whenever } x_1 y_2 \leq\ y_1 x_2$$

Note: This means that ordered pairs in R are of the form ((x_1, x_2), (y_1, y_2)).
Let $A\ =\ Z$, determine whether R satisfies each of the following properties. Justify your answers.

(i) Reflexive
(ii) Symmetric
(iii)Transitive

Now, Let $A\ =\ Z^{+}$, determine whether or not R is transitive. Justify your answer.

Solution

When $A\ =\ Z$,

(i) R is reflexive. Since $x_1 x_2 \leq\ x_1 x_2$, therefore $(x_1, x_2)\ R\ (x_1, x_2)$.

(ii) R is not symmetric. We can use a counterexample to prove that; $1 \times 2\ \leq\ 3 \times 1$, so $\big((1,1),(3,2)\big) \in R$. But $\big((3,2),(1,1)\big) \notin R$ since $3 \times 1\ >\ 1 \times 2$. There are of course many other examples that you could use.

(iii) R is not transitive**.** Let $(x_1, x_2) = (-1,-1), (y_1, y_2) = (2,3)$ and $(z_1, z_2) = (3,2)$. $\big((x_1, x_2),\ (y_1, y_2)\big) \in R$ since $(-1 \times 3)\ \leq\ (2 \times -1)$ and $\big((y_1, y_2),\ (z_1, z_2)\big) \in R$ since $(2 \times 2)\ \leq\ (3 \times 3)$. But $\big((x_1, x_2),\ (z_1, z_2)\big) \notin R$ since $(-1 \times 2)\ >\ (3 \times -1)$. There are other examples.

When $A\ =\ Z^{+}$, R is transitive.

Let $\big((x_1, x_2),\ (y_1, y_2)\big) \in R$ and $\big((y_1, y_2),\ (z_1, z_2)\big) \in R$, then

$x_1 y_2 \leq\ y_1 x_2$ and $y_1 z_2 \leq\ z_1 y_2$

$\frac{x_1}{x_2} \leq \frac{y_1}{y_2}$ and $\frac{y_1}{y_2} \leq \frac{z_1}{z_2}$ (This step is valid because $x_2, y_2, z_2 > 0$)

so $\frac{x_1}{x_2} \leq \frac{y_1}{y_2} \leq \frac{z_1}{z_2}$,

$\frac{x_1}{x_2} \leq \frac{z_1}{z_2} \rightarrow\ x_1 z_2 \leq\ z_1 x_2 \rightarrow \big((x_1, x_2),\ (z_1, z_2)\big) \in R$.

3) Let b(n) equal the value of the highest bit set to 1 in the binary representation of the positive integer n. (For example, b(27) = 16 because in 27 = 11011_2 and the most significant bit set to one is the first bit on the left, which has value 2^4.) Prove that the relation, R, defined below over the ***set of integers in between 1 and 1023, inclusive,*** is an equivalence relation. Into how many equivalence classes does R partition the set described? Explicitly list all of the members of the following equivalence classes: [2] and [13]. Let the set X be the largest of the equivalence classes. What is the smallest integer that belongs to X?

$$R = \{(x, y) \mid b(x) = b(y)\}$$

Solution

In order to prove that R is an equivalence relation, we must show that R is symmetric, reflexive and transitive.

For an integer x, the value of the highest bit in its binary representation is the same as itself which is why b(x) = b(x). Therefore, the (x, x) belongs in the relation R.

The relation R is symmetric as well. (x, y) has the binary representation b(x), b(y) and we know that b(x) = b(y). For (y, x), the same numbers x and y have the SAME binary representation so b(y) = b(x). Since both (x, y) and (y, x) belong to R, it is reflexive.

R is also transitive. If x and y have the same most significant bit as per b(x) = b(y), then for (y, z) the equation, b(y) = b(z), holds true. Since b(y) is equal to b(x), it follows that b(x) = b(z).

R = {(1,1), (2,2), (3,3), … , (2,1), (2,3), (4,5), (4,6), (4,7), (5, 4), (5, 6), (5,7), (6, 4), (6, 5), (6,7), ..)}

Binary Number	Decimal Number
0	0
1	1
10	2
11	3

[2] = {2,3}

1000	8
1001	9
1010	10
1011	11
1100	12
1101	13
1110	14
1111	15

[13] = {8, 9, 10, 11, 12, 13, 14, 15}

The largest equivalence class is [512], which has 512 members, all of which have their 9^{th} bit (worth 2^9) set to 1.

4) Let R be a relation over the positive integers defined as follows:

R = {(a,b) | gcd(a,b) > 1 but $a \nmid b$ and $b \nmid a$ }

In laymen's terms, describe how to determine whether or not two positive integers are related via R.

Determine whether or not R satisfies the following properties. Give a brief justification for each of your answers.

(i) reflexive
(ii) irreflexive
(iii) symmetric
(iv) anti-symmetric
(v) transitive

Two positive integers are related to one another via R if they share a common factor, but neither itself is a factor of the other.

(i) R is not reflexive. (a,a) ∉ R because a | a. (Violates condition)
(ii) Yes, it is irreflexive. The proof above shows that there are no positive integers a such that (a,a) ∈ R.
(iii) Yes, by definition, this relation is symmetric. If a and b are related, they share a common factor but not have one number be a factor of the other. If this is the case then b and a must ALSO share a common factor (the same one), and neither is a factor of the other.

(iv) No, note that (6, 15) ∈ R and (15, 6) ∈ R.
(v) No, note that (6, 15) ∈ R and (15, 35) ∈ R, but (6, 35) ∉ R.

5) How many anti-symmetric relations on the set A = {1, 2, 3, 4, 5, 6, 7} contain the ordered pairs (2, 3), (5, 2), (3, 3), (4, 4) and (6, 6)?

Solution

Anti-symmetric relationships can contain ordered pairs of the form (a, a), but don't "have" to. There are seven such ordered pairs, of which we are *forced* to include 3. We are free to do what we want with the other 4. There are 2^4 combinations of ways to include/not include ordered pairs of the form (a, a).

Anti-symmetric relationships are not allowed to contain two ordered pairs of the form (a, b) and (b, a) where a ≠ b. There are 7C2 = 21 such pairs of ordered pairs. Normally, for these ordered pairs, there are three possible arrangements: none or one of the ordered pairs are included in the relation. For 19 of the 21 pairs, we can make any of these 3 choices, for 3^{19} number of combinations of these ordered pairs. For the 20^{th} pair of {(5, 2), (2, 5)} and 21^{st} pair of {(2, 3), (3, 2)}, since we are forced to include (5, 2) and (2, 3) respectively, it forces us NOT to include (2, 5) and (3, 2), thus we have no freedom of choice for these pair of ordered pairs.

It follows that the desired number of relations is $2^4 3^{19}$.

6) Let f(x) = x^2 - 4x - 48 with a domain of all real $x \in (-\infty, 2]$. Prove that f is injective. What is the range of f? (You may either use Calculus or complete the square to prove your answers.)

Solution

To prove the function is injective, we must show that if f(a) = f(b), then a = b. For an arbitrary a and b, both in the interval $(-\infty, -2]$, let f(a) = f(b):

$$f(a) = f(b)$$

$$a^2 - 4a - 48 = b^2 - 4b - 48$$

$$a^2 - b^2 - 4a + 4b = 0$$

$$(a - b)(a + b) - 4(a - b) = 0$$

$$(a - b)(a + b - 4) = 0$$

Thus, we must have either $a - b = 0$ or $a + b - 4 = 0$. Remember that both a ≤ 2 and b ≤ 2. The only values of a and b that satisfy these inequalities AND satisfy the second equation are a = b = 2. (If either a or b is strictly less than 2, then the second equation CAN NOT be satisfied.) Notice that in this case a = b. Alternatively, a - b = 0, which ALSO means that a = b. Thus, we've proven that if f(a) = f(b), then a = b must follow. The range of f is [-52, ∞). We arrive at this by noting that x = 2 is a minimum of the function and before then, the function is continuous and decreasing. This is a minimum because f'(x) = 2x - 4. Setting f'(x) = 0 yields a turning point at x = 2.

7) Find $f^{-1}(x)$ for the function given in question #6.

Solution

$f(x) = x^2 - 4x - 48$

To find the inverse function, "swap" x and y and solve for y:

$x = y^2 - 4y - 48$

$x = y^2 - 2(2)(y) + 4 - 4 - 48$

$x = (y - 2)^2 - 52$

$x + 52 = (y - 2)^2$

When we take the square root of both sides, we must note that the domain of the original function was values of x less than or equal to 2. Thus, it now corresponds to the range of the inverse function. If we want y to be 2 or less, when we take the square root of both sides, we must choose the negative sign, thus for the known range of the inverse function, we must have:

$\sqrt{(y - 2)^2} = -\sqrt{x + 52}$

$y - 2 = -\sqrt{x + 52}$

$y = 2 - \sqrt{x + 52}$

It follows that the desired inverse function is $f^{-1}(x) = 2 - \sqrt{x + 52}$.

8) Let A be a set of 15 elements and B be a set of 12 elements. How many functions can be defined with the domain of A and the co-domain of B?

Solution

Each item in A must map to precisely 1 of 12 items in B. The mapping for a single item can be done in 12 ways, and each single item in A is independent of the rest. Thus, the total number of possible functions is 12^{15}, using the multiplication principle (repeatedly).

9) Let f(x) = $\sqrt{17x - x^2}$ and g(x) = 3x + 4, defined over a domain of [0, 17]. Determine f(g(x)) and g(f(x)).

Solution

$$f\big(g(x)\big) = f(3x+4) = \sqrt{17(3x+4) - (3x+4)^2}$$

$$= \sqrt{17(3x+4) - (3x+4)^2}$$

$$= \sqrt{51x + 68 - 9x^2 - 24x - 16}$$

$$= \sqrt{-9x^2 + 27x + 52}$$

$$g\big(f(x)\big) = g\big(\sqrt{17x - x^2}\big) = 3\big(\sqrt{17x - x^2}\big) + 4$$

10) Let f(x) = ax + b, where a and b are non-zero constants with a ≠ 1. Let $f^n(x)$ to be the function f composed with itself n times. (For example, $f^3(x)$ = f(f(f(x))).) Using trial and error, conjecture a guess for $f^n(x)$ and use mathematical induction to prove that guess. Your guess should be a closed form without any summations in it. The constants/variables that should appear in your guess are a, b, n and x.

Let's work out $f^2(x)$, $f^3(x)$ and $f^4(x)$ and see if we can find a pattern:

$$f^2(x) = f\big(f(x)\big) = f(ax+b) = a(ax+b) + b = a^2x + b(a+1)$$

$$f^3(x) = f(a^2x + b(a+1)) = a(a^2x + b(a+1)) + b = a^3x + b(a(a+1)+1)$$

$$f^4(x) = f\big(a^3x + b(a(a+1)+1)\big)$$

$$= a\big(a^3x + b(a(a+1)+1)\big) + b = a^4x + b(a(a(a+1)+1)+1)$$

It's clear that the coefficient to x is just a^n. Also, it's fairly easy to see the mechanism by how this works (we multiply the coefficient of x each time with the a...)

The other pattern for the constant term is more subtle. We can factor out a b from each of the constant terms (for f(x), $f^2(x)$, $f^3(x)$ and $f^4(x)$) to get the following sequence $1, (a+1)$, $a(a+1)+1, a(a(a+1)+1)+1$. If we now take the difference between each subsequent term we will get a, a^2+a+1, a^3+a^2+a+1 *and so on*. In other words, the terms mentioned will simply be equivalent to $\frac{a^n-1}{a-1}$. The constant term would then be equal to $b\frac{a^n-1}{a-1}$.

(Please note that the differences we obtain are determined by factorizing $x^n - 1$, however, since $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + .. + 1)$, we have to divide the term by x-1 to be left with $(x^{n-1} + x^{n-2} + .. + 1)$.

Thus, our conjecture is as follows: For all positive integers n, $f^n(x) = a^n x + b\frac{a^n-1}{a-1}$. We prove our conjecture via mathematical induction:

Base case: n = 1. LHS = $f^1(x) = ax + b$. RHS = $a^1x + b(a^1 - 1)/(a-1) = ax + b$.

Thus, the formula holds for the base case.

Inductive hypothesis: Assume for an arbitrary positive integer n = k that $f^k(x) = a^k x + b\frac{a^k-1}{a-1}$.

Inductive step: Prove for n = k + 1 that $f^{k+1}(x) = a^{k+1}x + b\frac{a^{k+1}-1}{a-1}$.

$f^{k+1}(x) = f(f^k(x))$

$= f(a^k x + b\frac{a^k-1}{a-1})$, using the inductive hypothesis

$= a\left(a^k x + b\frac{a^k-1}{a-1}\right) + b$, applying the function f.

$= a^{k+1}x + ab\left(\frac{a^k-1}{a-1}\right) + b$

$= a^{k+1}x + b(a\left(\frac{a^k-1}{a-1}\right) + 1)$

$= a^{k+1}x + b(\left(\frac{a^{k+1}-a}{a-1}\right) + 1)$

$= a^{k+1}x + b\left(\left(\frac{a^{k+1}-a+a-1}{a-1}\right)\right), after\ taking\ common\ denominator$

$= a^{k+1}x + b\left(\left(\frac{a^{k+1}-1}{a-1}\right)\right),$

This completes the inductive step. We can now conclude that for all positive integers n,

$$f^n(x) = a^n x + b\frac{a^n-1}{a-1}.$$

11) Give a summary of the life and mathematical contributions of Sophie Germain. Please aim for a length of roughly 200 - 400 words. **<u>Your summary must be typed.</u>** Please state the sources you used in writing your summary.

Sophie Germain, in full Marie-Sophie Germain, (born April 1, 1776, Paris, France—died June 27, 1831, Paris), French mathematician who contributed notably to the study of acoustics, elasticity, and the theory of numbers.

As a girl Germain read widely in her father's library and then later, using the pseudonym of M. Le Blanc, managed to obtain lecture notes for courses from the newly organized École Polytechnique in Paris. Germain's early work was in number theory, her interest having been stimulated by Adrien-Marie Legendre's Théorie des nombres (1789) and by Carl Friedrich Gauss's Disquisitiones Arithmeticae(1801).

In 1809 the French Academy of Sciences offered a prize for a mathematical account of the phenomena exhibited in experiments on vibrating plates conducted by the German physicist Ernst F.F. Chladni. Her third submitted memoir, with which she finally won the prize, treated vibrations of general curved as well as plane surfaces and was published privately in 1821. During the 1820s she worked on generalizations of her research but, isolated from the academic community on account of her gender and thus largely unaware of new developments taking place in the theory of elasticity, she made little real progress.

In 1819, Germain had actively revived her interest in number theory and wrote to Gauss outlining her strategy for a general solution to Fermat's last theorem, which states that there is no solution for the equation $x^n + y^n = z^n$ if n is an integer greater than 2 and x, y, and z are nonzero integers. She proved the special case in which x, y, z, and n are all relatively prime (have no common divisor except for 1) and n is a prime smaller than 100, although she did not publish her work. Her result first appeared in 1825 in a supplement to the second edition of Legendre's Théorie des nombres. She corresponded extensively with Legendre, and her method formed the basis for his proof of the theorem for the case n = 5. The theorem was proved for all cases by the English mathematician Andrew Wiles in 1995.

Germain continued to work in mathematics and philosophy until her death. Before her death, she outlined a philosophical essay which was published posthumously as *Considérations générales sur l'état des sciences et des lettres* in the *Oeuvres philosophiques*. She was stricken with breast cancer in 1829 but, undeterred by that and the fighting of the 1830 revolution, she completed papers on number theory and on the curvature of surfaces (1831).

References

http://www-groups.dcs.st-and.ac.uk/history/Biographies/Germain.html
https://www.britannica.com/biography/Sophie-Germain