# Homework #9

John Edwards

November 20, 2018

**Problem 1.** In a tennis league, there are 6 teams of 5 players each. A set of 7 players is selected randomly out of the 30 players in the league. What is the probability that at least one player from each team will be selected.

*Proof.* Our sample space, the total number of distinct 7 players sets, has $\binom{30}{7}$ elements. This means that one team will supply 2 players and the other teams will supply one player. There are 6 ways to choose the team that supplies two players. From that team, there are $\binom{5}{2}$, or 10 ways to select a pair of players. For the following 5 teams, we have 5 ways to select the single individual from that team. Each of these choices is independent and combined with the other, so we can count the total number of good sets by multiplying each of these values. This gives us $6 * 10 * 5^5$ number of sets with at least one player from each team. Thus, the final probability is: $\boxed{\frac{12 * 5^6}{\binom{30}{7}}}$. □

**Problem 2.** 30 tickets are sold in a raffle where 4 prizes will be given. Terri buys 3 of the tickets. What is the probability that Terri wins 0 prizes? 1 prize? 2 prizes? 3 prizes?

*Proof.* There are a total of $\binom{30}{4}$ drawings. In general, Terri can select k winning tickets in $\binom{3}{k}$ ways and she can select 3-k losing tickets in $\binom{26}{3-k}$ ways. The product of these two represents the total number of ways she selects exactly k winning tickets. Applying this formula for each value of k gives us the following: Terri wins 0 prizes with $\frac{\binom{26}{3}}{\binom{30}{4}}$ probability. Similarly, 1 prize can be won with probability $\frac{\binom{26}{2}\binom{4}{1}}{\binom{30}{4}}$, 2 with $\frac{\binom{26}{1}\binom{4}{2}}{\binom{30}{4}}$, and 3 with $\frac{\binom{4}{3}}{\binom{30}{4}}$. □

**Problem 3.** The integers from 1 to 10, inclusive, are partitioned at random into two sets of five elements each. What is the probability that 1 and 2 are in the same set?

*Proof.* The total number of pairs of sets of 5 integers is $\frac{\binom{10}{5}}{2!}$. Then to count the number of good sets place 1 and 2 together in the first set and choose the remaining 3 elements which can be done in $\binom{8}{3}$. So, the probability that 1 and 2 end up in the same set is $\frac{2\binom{8}{3}}{\binom{10}{5}}$. Simplifying we get $\frac{4}{9}$. More generally, we find that if we are given a set of 2n integers, the number of pairs of sets of n integers is $\frac{\binom{2n}{n}}{2!}$ and the number of sets with 1 and 2 together is $\binom{2n-2}{n-1}$.

If we plug in the definition of combinations and simplify the corresponding probability, we get the following:

$$\frac{\frac{2*(2n-2)!}{n!(n-2)!}}{\frac{2n!}{n!n!}} = \frac{2(2n-2)!n!n!}{(2n)!(n-2)!n!} = \frac{2n(n-1)}{(2n)(2n-1} = \frac{n-1}{2n-1}$$

A fundamentally easier way of looking at this problem is looking at each possible placement of teams as each permutation of 1 to 2n with 1 in the first slot and the first set being slots 1 to n. There are 2n-1 slots that the number 2 could be placed in, each equally likely. Of these, n-1 (numbered 2 through n) would put 2 in the same set as 1. Thus, this analysis also yields the correct answer of $\frac{n-1}{2n-1}$.

□

**Problem 4.** Sam's probability of getting an A on an individual test is 80 percent. If he takes ten tests, what is the probability he gets As on exactly 7 of those tests.

*Proof.* There are $\binom{10}{7}$ ways to choose which tests Sam passes, then the probability for each such choice of tests is $\binom{10}{7}(0.8)^7(0.2)^3$. □

**Problem 5.** You roll a pair of standard fair six-sided dice three times, and record the sum of the two faces showing each time. Let these three sums be $s_1, s_2, and s_3$. What is the probability that $s_3$ is strictly bigger than both $s_1$ and $s_2$.

*Proof.* For this problem it is rather straight forward to construct a table for the probability of each value of a sum. Then for each value of $s_3$ we know $s_1$ and $s_2$ must be strictly less than it. We can use our table to calculate this probability for each value of $s_3$ by summing the probability of each value strictly below $s_3$. Computing this sum we obtain $\sum_{k=2}^{11}(p(s_3 = k) * p(s_1 < k)^2 = \frac{1}{36}(2 * 1^2 + 3 * 3^2 + 4 * 6^2 + 5 * 10^2 + 6 * 15^2 + 5 * 21^2 + 4 * 26^2 + 3 * 30^2 + 2 * 33^2 + 1 * 35^2) = \frac{4345}{15552}$ □

**Problem 6.** Suppose we roll a fair 6-sided die with the numbers (1,...,6) written on them. After the first die roll we roll the die $k$ times where $k$ is the number on the first die roll. The number of points you score is the sum of the face-values on all die rolls (including the first). What is the expected number of points you will score.

*Proof.* The expected value of a single die roll is 3.5. Thus by linearity of expectation the expected value of $k$ dice rolls is $3.5k$. So remembering to account for the first die roll the expected value of this game is $\frac{1}{6}\sum_{k=1}^{6}(k + 3.5k) = 3.5 * 4.5 = \frac{63}{4}$. □

**Problem 7.** Suppose that one person in 1,000 people has a rare genetic disease. There is an excellent test for the disease; 99% of the people with the disease test positive and only 3% of the people who don't have it test positive. What is the probability that someone who tests positive has the disease? What is the probability that someone who tests negative does not have the disease?

*Proof.* Here D represents having the disease and !D not having it. Similarly, + represents testing positive and - represents testing negative.

Using Bayes' Theorem the answer to part one is:

$$P(D|+) = \frac{P(D \cap +)}{P(+)} = \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.03} = \boxed{0.032}$$

By the same method the answer to part two is:

$$P(!D|-) = \frac{P(!D \cap -)}{P(-)} = \frac{0.999 \times 0.97}{1 - (0.001 \times 0.99 + 0.03 \times 0.999)} = \boxed{0.999989}$$

□

**Problem 8.** Suppose $E$ and $F$ are events in a sample space and $p(E) = \frac{2}{3}$, $p(F) = \frac{3}{4}$, and $p(F|E) = \frac{9}{10}$. Find $p(E|F)$.

*Proof.* Using Bayes' Theorem

$$p(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(F|E)P(E)}{P(F)} = \frac{9 \times 2 \times 4}{10 \times 3 \times 3} = \boxed{\frac{4}{5}}.$$

□

**Problem 9.** Balls are randomly removed from a bag without replacement. If the probability that the first five balls withdrawn are all green is one-half, what is the fewest possible number of balls in the bag at the start?

*Proof.* Let the number of green balls be $G$ and the number of total balls $B$. Then, there are $\binom{G}{5}$ ways to choose the first five balls all green and $\binom{B}{5}$ total ways to choose 5 balls. So, the problem tells us that $\frac{\binom{G}{5}}{\binom{B}{5}} = \frac{1}{2}$. To solve this we write out the values of $\binom{N}{5}$ for $N = 5, 6, ..., 10$ and we find the $G = 9$ and $B = 10$ are the smallest possible values, thus the answer is 10. Another method with less computation is to realize that the minimum answer will involve $B$ total balls and $G = B$ -1 green balls. The corresponding probability of drawing five green balls in a row is $\frac{B-1}{B} \times \frac{B-2}{B-1} ... \times \frac{B-5}{B-4} = \frac{B-5}{B}$. Setting this to 1/2 yields $B = 10$. □

**Problem 10.** A Bubble Sort is a common algorithm taught to students that sorts a list of numbers. Given a random permutation of the integers 1,2,...,10, what is the probability that the permutation will be sorted with just one pass of Bubble Sort?

*Proof.* There are a total of 10! permutations. Consider running the algorithm backwards. Namely, start with a sorted array, and then either swap or don't swap. It's clear that all resulting arrays where you choose a swap will be different than all resulting arrays where you don't choose a swap. For example, for the last pair, if we don't swap it, the original permutation going backwards ends in a 10. Alternatively, if we do swap it, the original permutation going backwards doesn't end in a 10. Thus, we can count the total number of valid permutations by adding the permutations from each choice. In general though, for each potential swap, we have 2 choices, to swap or not to swap. There are n-1 potential swaps to consider, thus there are a total of $2^{n-1}$ permutations we can form going backwards that would be correctly sorted by one pass of Bubble Sort. So the probability of a permutation of length 10 being good is $\frac{2^9}{10!}$. To see this idea more concretely, consider the sequence (going backwards) of SNNNSSNNN, where we swap positions 9 and 10, don't swap the next three times, then swap positions 5 and 6, then swap positions 4 and 5, and then don't swap at all. Following these steps going backwards, we get the following state of the list:

1, 2, 3, 4, 5, 6, 7, 8, 10, 9
1, 2, 3, 4, 6, 5, 7, 8, 10, 9
1, 2, 3, 6, 4, 5, 7, 8, 10, 9

This last permutation corresponds to one that would get sorted with one pass of bubble sort. In particular, it swaps at positions 4, 5 and 9.

An alternate solution that is much more work is as follows. Notice that any sequence that could be sorted in one pass is a sequence that can be subdivided into separate pieces, where each piece is already sorted, and within a piece we would have to "move" a single larger number past all of its previous numbers, which are in order in front of it. For the example above, the pieces are [1], [2], [3], [6,4,5], [7], [8], [10, 9]. Any number of pieces is valid and each piece may have one or more numbers. Once the number of values in a single piece is set, the ordering of that piece is set. For example, if I tell you that the piece sizes are 1, 1, 1, 3, 1, 1, and 2. You can reconstruct the exact ordering given above. Thus, we can solve the problem by counting the number of ways to solve the equation $x_1 + x_2 + ..._{xk} = 10$ where each $x_i$ is a positive integer and k ranges from 1 to 10. Let's solve this for an arbitrary positive integer k, and then sum over all possible k. For an arbitrary positive integer k, we must subtract 1 for each of the k variables, leading us to count the number of solutions to $x_1 + x_2 + ..._{xk} = 10 - k$ where each $x_i$ is a non-negative integer. Using the formula for combinations with repetition, the number of solutions to this equation is $\binom{10-k+k-1}{k-1} = \binom{9}{k-1}$. If we sum this from k = 1 to k = 10, we get the sum of the binomial coefficients on row 9 of Pascal's Triangle, which is simply $2^9$, as desired. □

**Problem 11.** Andrew Wiles is a British number theorist credited with solving the long standing Fermat's Last Theorem. Reportedly, as a child he found the problem statement in a book at age 10 and was amazed that a problem which he could understand at his age had stumped professional mathematicians for centuries. Secretly, he set this aside and was very determined to solve it. Wiles, went on to study at Oxford and study elliptic curve theory, which luckily turned out useful in his solving of Fermat's Last Theorem. It is poor form to spend time attempting to solve such a problem when there is no visible, viable pathway to a solution. So Wiles was forced to put this problem aside for a while. Eventually Ken Ribet proved a conjecture which implied that proving a separate conjecture would imply Fermat's Last Theorem. At this point Wiles felt he could prove Fermat's Last Theorem. Thusly, he retreated into seclusion and worked on the problem alone for a period of 7 years. All the while releasing pre-written solutions to open problems so he would maintain his innocuous appearance. Then in 1993 he gave a three day talk, during the talks people began to see where it was going and on the third day the room was packed with mathematicians and photographers alike. Unfortunately an error was found in Wiles' original proof but was patched after a year of work (this time with collaborators). He received worldwide renown for putting the problem to rest, being knighted by the Queen of England and has since given many talks around the world and is now a research professor at Oxford.