

Fall 2018 COT 3100 Section 1 Homework 8 Solutions

1) Consider an ant that is walking on a Cartesian grid, starting at (0,0) and ending at (15, 18). The ant always chooses to walk exactly one unit either up or to the right (towards his destination) whenever he arrives at a Lattice point. (A Lattice point is a point with integer coordinates.) Thus, from (0,0) he either walks to (1, 0) or (0, 1). If the ant is not allowed to go to the points (6, 8) and (11, 15), how many different paths can he take on his walk?

Solution

Without any forbidden points, we know the ant must take 33 "steps" of which he must choose 15 to go to the right (positive x-axis). Thus, with the forbidden positions there are $\binom{33}{15}$ ways for the ant to walk the desired path. From this number, we must subtract out the number of forbidden paths.

The number of forbidden paths are the number of paths through (6, 8) or (11, 15). We can count these paths separately, but in doing so, we will have counted paths that go through both points twice, so we must subtract out all of those paths that go through both paths twice, via the Inclusion-Exclusion principle. In general, given a point (x, y) where $0 \leq x \leq 15$ and $0 \leq y \leq 18$, there are $\binom{x+y}{x}$ ways to get

to point (x, y) and $\binom{33-x-y}{15-x}$ ways to get from (x, y) to (15, 18). Since we can pair up any of the first set of paths with any of the second pair of paths, the product of these two terms is the total number of paths from (0, 0) to (15, 18) that go through (x, y). It follows that the number of paths through (6, 8) is

$\binom{14}{6}\binom{19}{9}$ and the number of paths through (11, 15) is $\binom{26}{11}\binom{7}{4}$. Finally, we must calculate the number of paths from (0, 0) to (15, 18) that go through both (6, 8) and (11, 15), since these were subtracted out twice and need to be added back in. All of these paths can be broken down into 3 sub-paths, ones that go from (0, 0) to (6, 8), then (6, 8) to (11, 15) and then from (11, 15) to (15, 18). We want to count the number of ways to take each of these subpaths and multiply all three of the terms. This yields: $\binom{14}{6}\binom{12}{5}\binom{7}{3}$ paths that go through both forbidden points, which now need to be added in. Our final result is:

$$\binom{33}{15} - \binom{14}{6}\binom{19}{9} - \binom{26}{11}\binom{7}{4} + \binom{14}{6}\binom{12}{5}\binom{7}{3}$$

2) This question considered permutations of "RONALDMCDONALD".

Solution

RONALDMCDONALD has 2 As, 1 C, 3 Ds, 2 Ls, 1 M, 2 Ns, 2 Os, and 1 R, so a total of 14 letters.

(a) How many permutations are there total?

There are $\frac{14!}{2!3!2!2!}$ permutations of these letters.

(b) How many permutations start and end with vowels?

Our vowels are 2 As, and 2 Os. We can have our permutation start and end with A, start and end with O, start with A end with O or start with O and end with A. Let's tackle the first two options. In these options, the 2 As or 2 Os are set. The ways to permute the remaining letters is $\frac{12!}{2!3!2!2!}$. Multiply this number by 2 to account for both cases, so we have $\frac{12!}{2!3!2!2!}$ ways where the permutations start and end with A or start and end with O. Now, let's say start with A and end with O. We still have 12 letters left to permute and can do

so in $\frac{12!}{2!3!2!}$ ways. Multiply this by 2 to cover both of these cases to get $\frac{12!}{2!3!}$ permutations that either start with A end with O or start with O end with A. Our final result is: $\frac{12!}{2!3!2!} + \frac{12!}{2!3!} = \frac{12!}{24} + \frac{12!}{12} = \frac{3(12!)}{24} = \frac{12!}{8}$.

(c) How many permutations do NOT have consecutive vowels in them?

Place our 10 consonants out with 11 gaps: _ C _ D _ D _ D _ L _ L _ M _ N _ N _ R _. We can choose 4 slots for our vowels out of the 11. Once we make that choice, we can permute the vowels is $\frac{4!}{2!2!}$ ways. Finally, we can permute our consonants in $\frac{10!}{3!2!2!}$ ways. Thus, the total number of permutations without consecutive vowels is $\binom{11}{4} \frac{4!10!}{2!2!3!2!2!} = \binom{11}{4} \frac{10!}{8}$.

(d) How many permutations contain the vowels in order (all As before all Os)?

There are 6 permutations of 2 As and 2 Os. Of these, only 1 has all of the vowels in order. Thus, if we count the total number of permutations, we can just divide this number by 6 to get the total number of permutations with the vowels in order. (To see this, note that for each permutation where the position of the vowels is fixed, there are 6 different orderings of the vowels, each of which is paired with each possible ordering of the consonants, since each of these are paired with the same thing, exactly $1/6^{\text{th}}$ of all permutations have the vowels in order.) Thus, there are a total of $\frac{14!}{2!3!2!2!2!6}$ permutations where the vowels are in order.

Note: an alternate way to solve this is to choose 4 slots out of 14 for the vowels, and then permute the consonants in $\frac{10!}{3!2!2!}$ ways. The vowels, once their slots are chosen, are fixed. Thus, the total number of permutations with the vowels in order is $\binom{14}{4} \frac{10!}{3!2!2!} = \frac{14!10!}{10!4!3!2!2!} = \frac{14!}{4!3!2!2!}$. A quick inspection of both denominators (from this answer and the previous one) shows that the two answers are the same number.

(e) How many permutations contain the substring "OLAND"? (Note: This one is hard!)

Let's make a super letter out of "OLAND". This leaves 1 A, 1 C, 2 Ds, 1 L, 1 M, 1 N, 1 O, and 1 R. There are 10 "letters" total here, with one letter, D, appearing twice. This gives us $\frac{10!}{2!}$ permutations with OLAND in them.

But...some permutations have been double counted because there are strings with 2 OLAND's in them. Every string with 2 OLAND's in them were counted twice, once when the super letter came first and once when the super letter came last. So, we must count the number of permutations that have 2 occurrences of OLAND and subtract this value out of our original answer since these were counted twice. So, if we have two super letters "OLAND" we are left with 1 C, 1 D, 1 M, and 1 R. Thus, we have 6 letters total with one repeated letter, "OLAND". There are $\frac{6!}{2!}$ such permutations.

Thus, the total number of permutations that contain the substring "OLAND" is $\frac{10!-6!}{2!} = \mathbf{1814040}$.

Note: To double check the answer, I wrote a python program. It's slow because Python is slow, but it does indeed verify the answer...

3) A class contains 18 girls and 14 boys. For all parts of this question, each boy and girl are distinguishable from one another. Answer the following questions:

Solution

(a) In how many ways can a committee of one boy and one girl be chosen?

We can choose a committee of one boy and one girl in $18 \times 14 = 252$ ways.

(b) In how many ways can a committee of five students be chosen?

We can choose a committee of 5 students in $\binom{32}{5}$ ways.

(c) In how many ways can a committee of four girls and three boys be chosen?

We can choose our 4 girls in $\binom{18}{4}$ ways and our boys in $\binom{14}{3}$ ways. Since we can pair up any choice of girls with boys, we multiply these to get a total of $\binom{18}{4} \binom{14}{3}$ possible committees.

(d) In how many ways can a committee of six students be chosen such that all the students on the committee are the same sex?

Just add the two options: all boys and all girls: $\binom{18}{6} + \binom{14}{6}$.

(e) In how many ways can the girls and boys form a line where no two boys are standing next to one another?

We place the girls in $18!$ ways, with 19 gaps. We then permute the boys in ${}_{19}P_{14}$ ways amongst the gaps. In factorials this is: $\frac{18!19!}{5!}$ orderings for the line. (Normally, we would choose 14 slots for the boys out of 19 and then permute the boys in $14!$ ways. This is equivalent to directly permuting the 14 boys in 19 slots.)

(f) How many committees of seven students contain at least two girls?

Count all committees of size 7 and then subtract out ones with 0 or 1 girls:

$$\binom{32}{7} - \binom{14}{7} - \binom{18}{1} \binom{14}{6}$$

4) How many solutions does the equation $a + b + c + d + e + f = 30$ have if each variable must be a non-negative integer and $a \leq 3$, $b \leq 7$ and $d \geq 8$?

Solution

If we force d to be 8 or greater, let's just create $d' = d + 8$ and restrict d' to be non-negative. Now, we count the number of solutions to:

$$a + b + c + d' + e + f = 22$$

with $a \leq 3$, $b \leq 7$.

Without the two restrictions there are $\binom{22 + 6 - 1}{6 - 1} = \binom{27}{5}$ solutions to the equation. Now, we'll subtract out the solutions where $a > 3$, then we'll subtract out the solutions where $b > 7$.

To count the number of solutions where $a > 3$, just set $a' = a + 4$ and find the number of non-negative solutions to $a' + b + c + d' + e + f = 18$. There are $\binom{18 + 6 - 1}{6 - 1} = \binom{23}{5}$ solutions to this equation.

To count the number of solutions where $b > 7$, just set $b' = b + 8$ and find the number of non-negative solutions to $a + b' + c + d' + e + f = 14$. There are $\binom{14 + 6 - 1}{6 - 1} = \binom{19}{5}$ solutions to this equation.

Finally, we double subtracted solutions where $a > 3$ AND $b > 7$. We must add these back in. To do this, set $a' = a + 4$ AND $b' = b + 8$ and find the number of solutions to $a' + b' + c + d' + e + f = 10$. There are $\binom{10 + 6 - 1}{6 - 1} = \binom{15}{5}$ solutions to this equation.

Putting together all of these pieces with the Inclusion/Exclusion Principle, we have a total of

$$\binom{27}{5} - \binom{23}{5} - \binom{19}{5} + \binom{15}{5}$$

solutions to the given equation that adhere to the restrictions give.

5) How many solutions does the equation $a + b + c + d + e + f + g + h \leq 40$ have if each variable must be a non-negative integer?

Solution

Let's create a new variable i , which is non-negative. If the sum of a through h is less than 40, i can simply carry the "slack" of this sum (difference from 40). So, we can get the answer to this question by finding the number of non-negative integer solutions to the equation:

$$a + b + c + d + e + f + g + h + i = 40$$

Basically, each solution to this equation in non-negative values is in one-to-one correspondence to a solution of the original equation with the inequality. (In some sense, the values of a through h "force" the value of i , so for each valid a through h , there is precisely one i that fits it in the second equation, and for every solution to the second equation it naturally maps to a single solution to the first equation.)

Thus, we can just count the number of solutions to this equation. There are $\binom{40 + 9 - 1}{9 - 1} = \binom{48}{8}$ such solutions.

6) A class has $2n$ students who must be split up into pairs. We consider two sets of pairs S and T different if at least one pair in the set S isn't a pair in the set T . A pair is unordered, so we consider the pair $(1, 2)$ and $(2, 1)$ to be the same pair. How many different sets of pairs can the class be split up into, in terms of n ? (For example, with $n = 2$, the answer is 3. Sets are $\{(1, 2), (3, 4)\}$, $\{(1, 3), (2, 4)\}$ and $\{(1, 4), (2, 3)\}$.)

Solution

Consider carrying out this process by always pairing up the lowest numbered student who isn't yet paired. At first, we'd have to pair up #1. There would be $2n - 1$ choices to pair up #1 with. Next, we'd either have to pair up #2 or #3. (If 2 was paired with 1, then we'd have to pair up 3 next. If 1 wasn't paired with 2, then we'd have to pair up 2 next.) Luckily, no matter which person we are pairing up, we are guaranteed that there are $2n-3$ choices for which to pair that person up with. For the third person to pair up, no matter who it is (could be 3, 4 or 5), we are guaranteed precisely $2n-5$ choices, and so forth. It follows that the solution is $\prod_{i=1}^n (2i - 1)$. There are many ways to express this quantity. One nice way to do so is $\frac{(2n)!}{2^n n!}$. We can derive this by noting that $(2n)!$ has all positive numbers from 1 to $2n$ multiplied together and from this we want to cross out the even values $2, 4, 6, \dots, 2n$. But, all of these even values can be written as a product of 2 and each integer from 1 to n . Regrouping, we get n copies of 2, and each integer from 1 to n in our product, which is just $n!$.

7) How many integers in between 1 and 10^7 are divisible by 15, 35, or 77?

Solution

Let set A be the set of integers in between 1 and 10^7 are divisible by 15.

Let set B be the set of integers in between 1 and 10^7 are divisible by 35.

Let set C be the set of integers in between 1 and 10^7 are divisible by 77.

Note that there are $\lfloor \frac{n}{k} \rfloor$ integers in between 1 and n , inclusive that are divisible by k , since if we group the integer in groups $(1, k), (k+1, 2k)$ and so forth, the last integer in each set is divisible by k .

If we want to know the number of integers divisible by both a and b , note that both a and b divide into $\text{lcm}(a, b)$ and both divide into any smaller number. Thus, if we want to answer the query for how many integers from 1 to n are divisible by both a and b , the result is $\lfloor \frac{n}{\text{lcm}(a,b)} \rfloor$.

Utilizing this logic and the Inclusion-Exclusion Principle, we have the number of integers in set A or set B or set C to be:

$$\left\lfloor \frac{10^7}{15} \right\rfloor + \left\lfloor \frac{10^7}{35} \right\rfloor + \left\lfloor \frac{10^7}{77} \right\rfloor - \left\lfloor \frac{10^7}{105} \right\rfloor - \left\lfloor \frac{10^7}{1155} \right\rfloor - \left\lfloor \frac{10^7}{385} \right\rfloor + \left\lfloor \frac{10^7}{1155} \right\rfloor = \mathbf{961038}$$

8) Give a summary of the life and mathematical contributions of Emmy Noether. Please aim for a length of roughly 200 - 400 words. **Your summary must be typed.** Please state the sources you used in writing your summary.

Sample Write Up

Emmy Noether was a German mathematician who lived towards the end of the 19th and beginning of the 20th centuries. In mathematics, she worked on Group Theory related objects such as rings and fields. She also made significant contributions to physics as well.

Emmy's father was a mathematician and though she was supposed to study French and English, she chose mathematics. After completing her degree at the University of Erlangen, where her father taught, she worked at the Mathematical Institute of Erlangen for several years (without pay). In 1915, about eight years after she got her degree, she was invited by David Hilbert to join the mathematics department at the University of Gottingen. Due to the era, other faculty objected and in order for her to teach courses, the courses were officially registered as being taught by Hilbert himself, even though Noether actually taught them.

Her early work included algebraic invariants and number fields and the theorem which bears her name, Noether's theorem is one of the most important theorems that helped guide physics in the early 20th century. Later, in her 1921 work, "Theory of Ideals in Ring Domains", she developed the idea of ideals in commutative rings, which turned out to be a very useful tool with many applications. Later she worked on noncommutative algebras and hypercomplex numbers.

Noether was well known for her excellence in Abstract Algebra and she was a part of the faculty at Gottingen (and recognized) through 1933, but then because Jews were getting removed from university positions, she relocated to the United States and taught at Bryn Mawr College. She ended up dying a couple years later from the complications of a surgery.

Source: https://en.wikipedia.org/wiki/Emmy_Noether