

COT 3100 Fall 2018 Homework #7 Solutions

1) Use induction on n to prove the following inequality for all positive integers n :

$$\sum_{i=1}^{2^n} \log_2 i \leq (n-1)2^n + 1$$

(Hint : Remember that $\log_2(2^x - y) \leq x$ when x is a positive integer and y is a non-negative integer such that $y < 2^x$.)

Solution

Base case: $n = 1$. LHS = $\sum_{i=1}^{2^1} \log_2 i = \log_2 1 + \log_2 2 = 0 + 1 = 1$
RHS = $(1-1)2^1 + 1 = 1$
Thus, the statement is true for $n = 1$.

Inductive hypothesis: Assume for an arbitrary positive integer $n = k$ that

$$\sum_{i=1}^{2^k} \log_2 i \leq (k-1)2^k + 1$$

Inductive step: Prove for $n = k+1$:

$$\sum_{i=1}^{2^{k+1}} \log_2 i \leq (k)2^{k+1} + 1$$

$$\sum_{i=1}^{2^{k+1}} \log_2 i = \sum_{i=1}^{2^k} \log_2 i + \sum_{i=2^k+1}^{2^{k+1}} \log_2 i$$

$$\leq (k-1)2^k + 1 + \sum_{i=2^k+1}^{2^{k+1}} \log_2 i, \text{ using the IH}$$

$$\leq (k-1)2^k + 1 + 2^k \log_2 2^{k+1}, \text{ since each of the } 2^k \text{ terms is } \leq \log_2 2^{k+1}$$

$$= (k-1)2^k + 1 + (k+1)2^k, \text{ using log power rule}$$

$$= (k-1+k+1)2^k + 1, \text{ combining terms}$$

$$= (2k)2^k + 1$$

$$= k2^{k+1} + 1, \text{ combining powers of 2}$$

This proves the inductive step. Thus, we've proven the original statement for all positive integers.

2) Define a recurrence relation t_n as follows: $t_0 = 1$, $t_1 = 5$, $t_n = 4t_{n-1} - 3t_{n-2}$, for all $n \geq 2$. Using induction on n , prove that $t_n = 2(3^n) - 1$.

Solution

Base case(s): $n = 0$ LHS = $t_0 = 2(3^0) - 1 = 2 - 1 = 1$, RHS = 1

$n = 1$ LHS = $t_1 = 2(3^1) - 1 = 6 - 1 = 5$, RHS = 5

The base case is true for both $n = 0$ and $n = 1$.

Inductive hypothesis: Let k be an arbitrarily chosen positive integer greater than or equal to 2. Assume for all non-negative integers $n \leq k$ that $t_n = 2(3^n) - 1$.

Inductive step: Prove for $n = k + 1$ that $t_{k+1} = 2(3^{k+1}) - 1$.

$t_{k+1} = 4t_k - 3t_{k-1}$, using given recurrence

$$= 4(2(3^k) - 1) - 3(2(3^{k-1}) - 1), \text{ plugging into IH for } k \text{ and } k-1$$

$$= 8(3^k) - 4 - 6(3^{k-1}) + 3$$

$$= 8(3^k) - 2(3)(3^{k-1}) - 1, \text{ using exponent power rule}$$

$$= 8(3^k) - 2(3^k) - 1$$

$$= 6(3^k) - 1, \text{ combining terms}$$

$$= 2(3)(3^k) - 1$$

$$= 2(3^{k+1}) - 1, \text{ using exponent power rule}$$

This proves the inductive step. Thus, we can conclude that the original statement is true for all non-negative integers.

3) Let H_n denote the n^{th} Harmonic number. (Recall that $H_n = \sum_{i=1}^n \frac{1}{i}$.)

Use mathematical induction on n to show that $H_{2^n} \geq 1 + \frac{n}{2}$, for all non-negative integers n .

Solution

Base case: $n = 0$. LHS = $H_{2^0} = H_1 = 1$. RHS = $1 + \frac{n}{2} = 1 + \frac{0}{2} = 1$. Since the two sides are equal, the given formula is true for $n = 0$.

Inductive hypothesis: Assume for an arbitrarily chosen non-negative integer $n = k$ that $H_{2^k} \geq 1 + \frac{k}{2}$.

Inductive step: Prove for $n = k+1$ that $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$.

$$\begin{aligned}
 H_{2^{k+1}} &= \sum_{i=1}^{2^{k+1}} \frac{1}{i} \\
 &= \left(\sum_{i=1}^{2^k} \frac{1}{i} \right) + \left(\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \right) \\
 &= H_{2^k} + \left(\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \right) \\
 &\geq 1 + \frac{k}{2} + \left(\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \right) \\
 &\geq 1 + \frac{k}{2} + \left(\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{2^{k+1}} \right) \\
 &= 1 + \frac{k}{2} + \left(\frac{1}{2^{k+1}} \sum_{i=2^k+1}^{2^{k+1}} 1 \right) \\
 &= 1 + \frac{k}{2} + \left(\frac{2^k}{2^{k+1}} \right) \\
 &= 1 + \frac{k}{2} + \left(\frac{1}{2} \right) \\
 &= 1 + \frac{k+1}{2}
 \end{aligned}$$

This completes the inductive step. Thus, we can conclude for all non-negative integers n , $H_{2^n} \geq 1 + \frac{n}{2}$.

4) Let $f(n) = \frac{n}{n+2}$. Define $f^k(n)$ to be the function f composed with itself k times. More formally, $f^0(n) = n$ and $f^k(n) = f(f^{k-1}(n))$, for all positive integers k . Using induction on k , prove that for all positive integers k , $f^k(n) = \frac{n}{(2^k-1)n+2^k}$. (Hint: The algebra can be messy if you don't multiply both your numerator and denominator by $(2^k - 1)n + 2^k$. So, in full, after you do a particular step, you would take your fraction and multiply it by $\frac{(2^k-1)n+2^k}{(2^k-1)n+2^k}$. Please feel free to ignore the hint, but I do think it reduces the amount of algebra drastically.) (Note: This is an exam question from a previous semester, so a good question for practice!)

Solution

Base case: $k = 1$. LHS = $f^1(n) = f(f^0(n)) = f(n) = \frac{n}{n+2}$

$$\text{RHS} = \frac{n}{(2^1-1)n+2^1} = \frac{n}{n+2}$$

For all values of n , these two functions are equal, so the base case holds.

Inductive Hypothesis: Assume for an arbitrarily chosen positive integer $k = m$ that

$$f^m(n) = \frac{n}{(2^m-1)n+2^m}.$$

Inductive Step: Prove for $k = m+1$ that $f^{m+1}(n) = \frac{n}{(2^{m+1}-1)n+2^{m+1}}$

$$\begin{aligned} f^{m+1}(n) &= f(f^m(n)) \\ &= f\left(\frac{n}{(2^m-1)n+2^m}\right), \text{ using I. H.} \\ &= \frac{\frac{n}{(2^m-1)n+2^m}}{\frac{n}{(2^m-1)n+2^m} + 2} \\ &= \frac{\frac{n}{(2^m-1)n+2^m}}{\frac{n}{(2^m-1)n+2^m} + 2} \times \frac{(2^m-1)n+2^m}{(2^m-1)n+2^m} \\ &= \frac{n}{n+2((2^m-1)n+2^m)} \\ &= \frac{n}{n+(2^{m+1}-2)n+2^{m+1}} \\ &= \frac{n}{(2^{m+1}-1)n+2^{m+1}} \end{aligned}$$

This completes the inductive step. Thus, we can conclude for all positive integers k ,

$$f^k(n) = \frac{n}{(2^k-1)n+2^k}.$$

5) Give a summary of the life and mathematical contributions of Paul Erdos. Please aim for a length of roughly 200 - 400 words. **Your summary must be typed.** Please state the sources you used in writing your summary.

Sample Write Up

Paul Erdos was a prolific Hungarian mathematician who lived entirely in the twentieth century (1913 - 1996). Much like Kevin Bacon is known for acting with many different people, Erdos is known for having collaborated with many diverse mathematicians on academic papers (over 500 people!) In fact, many academics calculate their own Erdos number. Erdos is the only person to have an Erdos number of 0. All of his direct collaborators have Erdos number 1. Everyone who collaborated with those individuals (but not Erdos) have an Erdos number of 2, and so forth.

He proposed and solved problems in many different fields of mathematics but is best known for his work in discrete mathematics. He most enjoyed attempting to solve open problems that others had posed instead of creating new branches of mathematics. Erdos had a turbulent childhood due to his father being imprisoned and his mother having to work long hours, but found solace in mathematics. At the young age of 21, he earned his Ph.D. from the University of Budapest. Unlike most, at most points in his life, he never had a single home. Erdos loved traveling and collaborating on mathematics. There are many stories of Erdos showing up at a colleagues doorstep, asking to stay for a few days while they worked on problems. In fact, for a few days in the 1980s, Erdos stayed with Dr. Ronald Dutton, a former professor in the Computer Science department at UCF!

Erdos is well-known for both consuming great quantities of coffee and amphetamines. In fact, one of his colleagues, Ron Graham, bet him \$500 that he couldn't stop taking the amphetamines for a month. Erdos won the bet but then complained that mathematical progress had been set back a month because he couldn't make much progress without the amphetamines!

Mathematically, Erdos developed Ramsey theory and made advances in extremal combinatorics. He found a proof of Bertrand's Postulate and a much easier proof of the prime number theorem (about the distribution of prime numbers) than the previous proof. In addition, he would offer rewards for solutions to unsolved problems. Prizes were as little as \$25 up to a few thousand dollars. His friend Ron Graham, who made the amphetamine bet, was the administrator of awarding the prizes.

Erdos passed away at a conference in Warsaw at the age of 83, only after publishing more than 1500 mathematical papers, more than any one in history except Leonard Euler.

Sources: https://en.wikipedia.org/wiki/Paul_Erd%C5%91s and
Conversation with Ron Dutton