

COT 3100 Fall 2018 Homework #6 Solutions

1) Use mathematical induction on n to prove the following assertion for all positive integers n :
$$\sum_{i=1}^n \frac{i(i+1)}{2} = \frac{n(n+1)(n+2)}{6}.$$

Solution

Base case: $n = 1$. LHS = $\sum_{i=1}^1 \frac{i(i+1)}{2} = \frac{1(2)}{2} = 1$, RHS = $\frac{1(1+1)(1+2)}{6} = \frac{6}{6} = 1$. It follows that the given statement is true for $n = 1$ and the base case holds.

Inductive hypothesis: Assume for an arbitrary positive integer $= k$ that

$$\sum_{i=1}^k \frac{i(i+1)}{2} = \frac{k(k+1)(k+2)}{6}$$

Inductive step: Prove for $n = k + 1$ that

$$\sum_{i=1}^{k+1} \frac{i(i+1)}{2} = \frac{(k+1)(k+2)(k+3)}{6}$$

$$\begin{aligned} \sum_{i=1}^{k+1} \frac{i(i+1)}{2} &= \sum_{i=1}^k \frac{i(i+1)}{2} + \frac{(k+1)(k+2)}{2} \\ &= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} \\ &= \frac{k(k+1)(k+2)}{6} + \frac{3(k+1)(k+2)}{6} \\ &= \frac{(k+1)(k+2)[k+3]}{6} \end{aligned}$$

This proves the inductive step, as desired. (Note: the inductive hypothesis was used on the second line of the proof, when substituting for the summation from $i=1$ to k .)

2) Use induction on n to prove that $4^{2n} - 15n - 1$ is divisible by 225 for all non-negative integers n .

Solution

Base case: $n=0$. The expression evaluates to $4^{2(0)} - 15(0) - 1 = 0$, which is divisible by 225

Inductive hypothesis: Assume for an arbitrary integer $n=k$ that $4^{2k} - 15k - 1$ is divisible by 225. Namely, assume that $4^{2k} - 15k - 1 = 225a$ for some integer a .

Inductive step: Prove for $n=k+1$ that $4^{2(k+1)} - 15(k+1) - 1$ is divisible by 225. Namely, show that $4^{2(k+1)} - 15(k+1) - 1 = 225b$, for some integer b .

$$\begin{aligned} 4^{2(k+1)} - 15(k+1) - 1 &= 4^{2k+2} - 15k - 15 - 1 \\ &= 4^2 4^{2k} - 15k - 16 \\ &= 16(4^{2k}) - 15k - 16 \\ &= 16(4^{2k}) - (16)15k - 16 + 15(15k), \text{ add/subtracting } 15(15k) \\ &= 16(4^{2k} - 15k - 1) + 225, \text{ factoring out } 16 \text{ from the first 3 terms} \\ &= 16(225a) + 225, \text{ using the inductive hypothesis} \\ &= 225(16a+1), \end{aligned}$$

Since a is an integer, let $b = 16a + 1$, and we are done. Thus, for all non negative integers n , $4^{2n} - 15n - 1$ is divisible by 225.

3) Use induction to show that $\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -2^n + 1 & 2^n \end{pmatrix}$ for all positive integers n.

Solution

Base case: n=1 LHS = $\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$, RHS = $\begin{pmatrix} 1 & 0 \\ -2^1 + 1 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$,
so the base case is true. (1 pts)

Inductive hypothesis: Assume for an arbitrary integer n=k that $\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ -2^k + 1 & 2^k \end{pmatrix}$. (1 pt)

Inductive step: Prove for n=k+1 that $\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ -2^{k+1} + 1 & 2^{k+1} \end{pmatrix}$ (1 pts)

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \quad (2 \text{ pts}) \\ &= \begin{pmatrix} 1 & 0 \\ -2^k + 1 & 2^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, \text{ using the inductive hypothesis (2 pts)} \\ &= \begin{pmatrix} 1 & 0 \\ -2^k + 1 - 2^k & 2(2^k) \end{pmatrix}, \text{ multiplying out the matrices (1 pt)} \\ &= \begin{pmatrix} 1 & 0 \\ -2^{k+1} + 1 & 2^{k+1} \end{pmatrix}, \text{ using exponent rules for simplification. (2 pt)} \end{aligned}$$

Thus, it follows that the assertion is true for all positive integers n.

Sample Summary

Carl Friedrich Gauss is known as one of the world's best mathematicians. Born in present day Germany in 1777, Gauss showed talent in mathematics very early on. Anecdotally, he is given credit for discovering how to add the first hundred integers at the age of 8. By the age of 21, he had produced a ground breaking work on Number Theory, Disquisitiones Arithmeticae. One of Gauss's early triumphs was determining exactly which regular polygons could and could not be constructed with a compass and a straight edge. Though this appears to be a geometry problem, the ultimate solution heavily involves number theory as he found that the construction is possible if and only if the number of sides is a product of distinct Fermat primes and a power of 2. (Note: A Fermat prime is a prime number of the form $2^k + 1$, where k is a positive integer. It turns out that number of this form may be prime only if k itself is a power of 2. There are only 5 known Fermat primes.) Another notable result of Gauss's was determining an efficient method for calculating which values are quadratic residues. (A value q is a quadratic residue mod n if and only if there is an integer x such that $x^2 \equiv q \pmod{n}$.) The rule that is used to make this calculation is Gauss's Law of Quadratic Reciprocity. Later in Gauss's career, he studied fields other than mathematics. Outside of mathematics, he's best known for his contribution to magnetism with the law that is simply now known as Gauss's Law. The law states that, "the area integral of the electric field over any closed surface is equal to the net charge enclosed in the surface divided by the permittivity of space." Gauss's Law is one of four equations known as Maxwell's Equations, which are the key cornerstone to any physics course in electricity and magnetism.

Sources

https://en.wikipedia.org/wiki/Carl_Friedrich_Gauss

<http://hyperphysics.phy-astr.gsu.edu/hbase/electric/gaulaw.html>