

Homework5

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1) An arithmetic sequence a_1, a_2, a_3, \dots is such that $a_{10} = 40$ and $a_{25} = 10$. Determine both a_1 and the sum of the first 50 terms of the sequence.

Since arithmetic sequence, we have:

$$a_j = a_i + (j - i)d$$

Then,

$$d = \frac{a_j - a_i}{j - i}$$

Plug in the a_{10} and a_{25} :

$$d = \frac{a_{25} - a_{10}}{25 - 10} = \frac{10 - 40}{15} = -2$$

Also,

$$a_1 = a_{10} + (1 - 10)d$$

We can get:

$$a_1 = 40 + (-9) \times (-2) = 58$$

For sum of arithmetic sequence:

$$\begin{aligned} S_n &= \frac{(a_1 + a_n) \times n}{2} \\ S_{50} &= \frac{(a_1 + a_{50}) \times 50}{2} \\ &= \frac{(a_1 + a_1 + 49 \times d) \times 50}{2} \\ &= \frac{(58 + 58 + 49 \times (-2)) \times 50}{2} \\ &= 450 \end{aligned}$$

2) An infinite geometric sequence a_1, a_2, a_3, \dots has a sum of 10. If the odd indexed terms ($a_1, a_3, \text{etc.}$) have a sum of 7, what is the common ratio of the sequence? What is the first term of the sequence?

Since **infinite** geometric sequence, we assume r is the common ratio, then we have:

$$S_n = \frac{a_1}{1-r} = 10$$

For all odd indexed terms, they still form a infinite geometric sequence and it takes r^2 as the ratio, then we have:

$$S_{\text{odd}_n} = \frac{a_1}{1-r^2} = 7$$

Then, solve the system of equations:

$$\begin{cases} \frac{a_1}{1-r} = 10 \\ \frac{a_1}{1-r^2} = 7 \end{cases}$$

we get:

$$\begin{cases} a_1 = \frac{40}{7} \\ r = \frac{3}{7} \end{cases}$$

3) Determine the following summation in terms of n : $\sum_{i=1}^{2n} (3i^2 - 4i)$.

$$\begin{aligned} \sum_{i=1}^{2n} (3i^2 - 4i) &= \sum_{i=1}^{2n} 3i^2 - \sum_{i=1}^{2n} 4i \\ &= 3 \sum_{i=1}^{2n} i^2 - 4 \sum_{i=1}^{2n} i \end{aligned}$$

From test 1 formula sheet:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

We can derive:

$$\sum_{i=1}^{2n} i^2 = \frac{(2n)(2n+1)(2 \times 2n+1)}{6} = \frac{n(2n+1)(4n+1)}{3}$$

Also:

$$\sum_{i=1}^{2n} i = \frac{(1+2n)2n}{2} = (1+2n)n$$

Then:

$$\begin{aligned}\sum_{i=1}^{2n} (3i^2 - 4i) &= n(2n+1)(4n+1) - 4(1+2n)n \\ &= 8n^3 - 2n^2 - 3n\end{aligned}$$

4) Determine the following infinite summation: $\sum_{i=1}^{\infty} (2i-1)\left(\frac{1}{2}\right)^i$

Notice that the each term in the summation is built by a term of arithmetic sequence multiply a term of geometry sequence, we can try to multiply the ratio of the geometry sequence to the original summation to form a new summation and then subtract the original summation from the new summation.

Assume the result of infinite summation is p ,

$$\begin{aligned}p &= \sum_{i=1}^{\infty} (2i-1)\left(\frac{1}{2}\right)^i \\ p &= 1 \times \left(\frac{1}{2}\right)^1 + 3 \times \left(\frac{1}{2}\right)^2 + 5 \times \left(\frac{1}{2}\right)^3 + \dots + (2n-1) \times \left(\frac{1}{2}\right)^n + \dots\end{aligned}$$

Now multiply the ratio $\frac{1}{2}$ to both sides to form a new summation:

$$\frac{1}{2}p = 1 \times \left(\frac{1}{2}\right)^2 + 3 \times \left(\frac{1}{2}\right)^3 + 5 \times \left(\frac{1}{2}\right)^4 + \dots + (2n-3) \times \left(\frac{1}{2}\right)^n + (2n-1) \times \left(\frac{1}{2}\right)^{n+1} + \dots$$

Subtract,

$$p - \frac{1}{2}p = 1 \times \left(\frac{1}{2}\right)^1 + (3-1) \times \left(\frac{1}{2}\right)^2 + (5-3) \times \left(\frac{1}{2}\right)^3 + \dots + ((2n-1)-(2n-3)) \times \left(\frac{1}{2}\right)^n + \dots$$

Then,

$$\frac{1}{2}p = \frac{1}{2} + 2 \times \left(\frac{1}{2}\right)^2 + 2 \times \left(\frac{1}{2}\right)^3 + \dots + 2 \times \left(\frac{1}{2}\right)^n + \dots$$

Distributive 2,

$$\frac{1}{2}p = \frac{1}{2} + 2 \times \left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots + \left(\frac{1}{2}\right)^n + \dots\right)$$

Apply sum of infinite geometry sequence,

$$\frac{1}{2}p = \frac{1}{2} + 2 \times \frac{\left(\frac{1}{2}\right)^2}{1 - \frac{1}{2}}$$

Tada:

$$p = 3$$

$$\sum_{i=1}^{\infty} (2i - 1) \left(\frac{1}{2}\right)^i = 3$$

5) Determine the following matrix sum:

$$\begin{bmatrix} 6 & 3 & -2 \\ 5 & -9 & 7 \end{bmatrix} + \begin{bmatrix} -2 & 12 & 3 \\ 6 & 4 & 8 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 6 & 3 & -2 \\ 5 & -9 & 7 \end{bmatrix} + \begin{bmatrix} -2 & 12 & 3 \\ 6 & 4 & 8 \end{bmatrix} &= \begin{bmatrix} 6-2 & 3+12 & -2+3 \\ 5+6 & -9+4 & 7+8 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 15 & 1 \\ 11 & -5 & 15 \end{bmatrix} \end{aligned}$$

6) Determine the following matrix product,

$$\begin{bmatrix} 2^n & 1 - 2^n \\ 3 & 2^{n-1} \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$$

, in terms of n.

$$\begin{aligned} \begin{bmatrix} 2^n & 1 - 2^n \\ 3 & 2^{n-1} \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} &= \begin{bmatrix} 2^n \times 3 + 1 - 2^n & 2^n \times 2 - 1 + 2^n \\ 3 \times 3 + 2^{n-1} & 3 \times 2 - 2^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} 2^{n+1} + 1 & 2^n - 1 \\ 9 + 2^{n-1} & 6 - 2^{n-1} \end{bmatrix} \end{aligned}$$

7) Let the Fibonacci Sequence be defined as follows: $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$, for all integers $n \geq 2$. Determine and simplify the following matrix product, in terms of n:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} & F_{n-1} + F_{n-2} \\ F_n - F_{n-1} & F_{n-1} - F_{n-2} \end{bmatrix}$$

Since $F_n = F_{n-1} + F_{n-2}$, then $F_n + F_{n-1} = F_{n+1}$, $F_{n-1} + F_{n-2} = F_n$,
 $F_n - F_{n-1} = F_{n-2}$, $F_{n-1} - F_{n-2} = F_{n-3}$ where $n \geq 3$
when $n = 2$,

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} &= \begin{bmatrix} F_2 + F_1 & F_1 + F_0 \\ F_2 - F_1 & F_1 - F_0 \end{bmatrix} \\ &= \begin{bmatrix} F_1 + F_0 + F_1 & F_1 + F_0 \\ F_0 & F_1 - F_0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Then,

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \text{ where } n = 2 \\ &= \begin{bmatrix} F_{n+1} & F_n \\ F_{n-2} & F_{n-3} \end{bmatrix}, \text{ where } n \geq 3 \end{aligned}$$