Homework #3

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Problem 1. In general, you were told in class that for all integers a and b and positive integers n, if $a \equiv b \mod n$, then $f(a) \equiv f(b) \mod n$, where f is any function that operates on integers only. Using the definition of mod only, prove this specifically for the function $f(a) = a^3$.

Proof. If $a \equiv b \mod n$, then $b = a + nk$ for some integer k. So $f(a) \equiv a^3 \mod n$ and $f(b) \equiv b^3 \equiv (a+nk)^3 \equiv a^3 + 3a^2(nk) + 3a(nk)^2 + (nk)^3 \equiv a^3 \equiv f(a) \mod n$. □

Problem 2. Convert the following values from the bases indicated to base 10:

Proof.
I) $2165_7 = 2*7^3 + 1*7^2 + 6*7^1 + 5*7^0 = 782$
II) $\text{BCF}2_16 = 11*16^3 + 12*16^2 + 15*16^1 + 2*16^0 = 48370$
III) $12345_8 = 1*8^4 + 2*8^3 + 3*8^2 + 4*8^1 + 5*8^0 = 5349$
IV) $21302_4 = 2*4^4 + 1*4^3 + 3*4^2 + 0*4^1 + 2*4^0 = 626$
V) $101001111101_2 = 1*2^{11} + 1*2^9 + 1*2^6 + 1*2^5 + 1*2^4 + 1*2^3 + 1*2^2 + 1*2^0 = 2685$ □

Problem 3. Convert the following base 10 values to the bases indicated:

Proof.
I) $22111 = 7*3158 + 5$
$3158 = 7*451 + 1$
$451 = 7*64 + 3$
$64 = 7*9 + 1$
$9 = 7*1 + 2$
$1 = 7*0 + 1$
So the answer is 121315_7.

II) $83810 = 16*5238 + 2$
$5238 = 16*327 + 6$
$327 = 16*20 + 7$
$20 = 16*1 + 4$
$1 = 16*0 + 1$
So the answer is 14762_{16}.

III) $907 = 2 * 453 + 1$
$453 = 2 * 226 + 1$
$226 = 2 * 113 + 0$
$113 = 2 * 56 + 1$
$56 = 2 * 28 + 0$
$28 = 2 * 14 + 0$
$14 = 2 * 7 + 0$
$7 = 2 * 3 + 1$
$3 = 2 * 1 + 1$
$1 = 2 * 0 + 1$
So the answer is 1110001011_2.

IV) $3209 = 7 * 458 + 3$
$458 = 7 * 65 + 3$
$65 = 7 * 9 + 2$
$9 = 7 * 1 + 2$
$1 = 7 * 0 + 1$
So the answer is 12233_7.

V) $4095 = 8 * 511 + 7$
$511 = 8 * 63 + 7$
$63 = 8 * 7 + 7$
$7 = 8 * 0 + 7$
So the answer is 7777_8. □

Problem 4. Prove or disprove: if $n(3n + 1)$ is an even integer, then n is an integer.

Proof. Counterexample: If $n(3n + 1) = 0$ then we can have $n = -1/3$ so the statement is false. □

Problem 5. A common divisibility rule is that a positive integer $n = d_k d_{k-1} ... d_0$, is divisible by 9 if and only if $d_k + d_{k-1} + ... + d_0$ is divisible by 9. Rigorously prove this divisibility rule via mod rules.

Proof. If n is divisible by 9 then we can look at the base 10 representation of n. So, $n \equiv d_k 10^k + d_{k-1} 10^{k-1} + ... + d_0 10^0 \mod 9$. Now, for every power of 10 we know that $10^k \equiv (10 \mod 9)^{k-1} \mod 9$, so $10^k \equiv 1 \mod 9$. Thus, $d_k 10^k + d_{k-1} 10^{k-1} + ... + d_0 10^0 \equiv d_k + d_{k-1} + ... + d_0 \mod 9$. So, if $n \equiv 0 \mod 9$, then $d_k + d_{k-1} + ... + d_0 \equiv 0 \mod 9$. In the other direction we can add $d_i(10^i - 1)$ to d_i in our sum for each i, this will get us n and since $10^i - 1 \equiv 0 \mod 9$ we will have $d_k + d_{k-1} + ... + d_0 \equiv n \equiv 0 \mod 9$. □

Problem 6. Let x and y be integers such that $13|(2x + 3y)$. Prove that $13|(31x + 27y)$.

Proof. If $13|(2x + 3y)$, then 13 must also divide $9 * (2x + 3y) = 18x + 27y$. Also notice that for any integer x it is clear that $13|13x$. So, 13 must divide the sum of these two expressions. That is to say, $13|(31x + 27y)$. □

Problem 7. Give a summary of the life and mathematical contributions of Pierre de Fermat. Please aim for a length of roughly 200-400 words. Your summary must be typed. Please state the sources you used in writing your summary.

Pierre de Fermat was born late 1607 to a wealthy leather merchant. From 1623 to 1626 Fermat attended the University of Orleans where he obtained his bachelor in civil law. A little after getting his degree Fermat had his first encounter with mathematics, the field he's best known for. Among other things Fermat is well known for his contributions to the field of number theory, famously he discovered Fermats Little Theorem which states that if an integer a is coprime to a prime p then $a^{p-1} \equiv 1 \mod p$. Perhaps the thing he is most well known for though, is his famous Last Theorem. In the margins of a book he wrote down that he had a proof of the fact that there were no solutions to $a^n + b^n = c^n$ where a, b, c are positive integers and n is greater than 2, but it was too small to fit in the margin of the book he was writing in. This problem stood untouched for approximately 300 years until progress was made on a few key conjectures which led to the final solution by Andrew Wiles in 1994. Fermat also worked with Blaise Pascal to help lay the foundation of probability theory. Independently Fermat laid foundation for the development of calculus by Newton and Leibniz through his study of minima and maxima. After his death in 1665, many of notebooks were shown to the public and contained a number of interesting problems i.e. Fermats Last Theorem.