Homework 2

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1) For all integers n, prove that $n(3n + 1)$ is an even integer.

Case 1: If n is an even integer, let $n = 2k$, where $k \in Z$

$$\begin{aligned} n(3n+1) &= (2k) \times [3 \times (2k) + 1] \\ &= 2 \times (k \times (3 \times (2k) + 1)) \end{aligned}$$

Case 2: If n is an odd integer, let $n = 2k + 1$, where $k \in Z$

$$\begin{aligned} n(3n+1) &= (2k+1) \times (3 \times (2k+1) + 1) \\ &= (2k+1) \times (6k+4) \\ &= 2 \times ((2k+1) \times (3k+2)) \end{aligned}$$

By the definition of even integer, $2 \times (k \times (3 \times (2k) + 1))$ and $2 \times ((2k + 1) \times (3k + 2))$ are both even integers where $k \in Z$, proved!

2) Let n be an odd integer. Using the result from question #1, prove that $16|(n^4 - 1)$. (Note: this question does require you to multiply out an expression of the form $(x + y)^4$. The algebra is not as bad as it looksalso you can apply the binomial theorem to speed up the work. Google it if you haven't seen it before.)

Since n is an odd integer, let $n = 2k + 1$, where $k \in Z$.

$$\begin{aligned} (n^4 - 1) &= (2k+1)^4 - 1 \\ &= (4k^2 + 4k + 1)^2 - 1 \\ &= (16k^4 + 16k^2 + 1 + 32k^3 + 8k^2 + 8k) - 1 \\ &= (16k^4 + 32k^3) + (24k^2 + 8k) \\ &= 16(k^4 + 2k^3) + 8k(3k+1) \end{aligned}$$

Plugin the result from #1, $k(3k+1)$ is even, which means $k(3k+1)$ is divisible by 2, then $8k(3k+1)$ is divisible by 16. Also $16(k^4+2k^3)$ is divisible by 16, then we can get that $16(k^4+2k^3)+8k(3k+1)$ is divisible by 16, proved!

3) Given a set of n positive real numbers $a_1, a_2, , a_n$, with an average of b, prove that the value of the largest element in the set is strictly less than bn.

Proof by contradiction: let p be "The average of a set of real positive numbers $a_1, a_2, \ldots, a_n$, is b" and q be "The largest element in the set is strictly less than bn". To construct a proof by contradiction, assume that both p and $\neg q$ are true. That is, assume that the largest element in the set is greater and equal than bn, then the sum of set will be:

$$sum = a_{largest} + \text{sum of rest elements in set } a$$

Then the average of set a will be:

$$
\begin{aligned}
average &= \frac{a_{largest} + \text{sum of rest elements in set } a}{n} \\
&>= \frac{bn + \text{sum of rest elements in set } a}{n} \\
&>= b + \frac{\text{sum of rest elements in set } a}{n}
\end{aligned}
$$

Since set a contains positive real numbers, then $\frac{\text{sum of rest elements in set } a}{n} > 0$, $average > b$ which means p is false. Since p is both true and false, then we have on contradiction here. Value of the $\max(a_i)$ in the set is strictly less than bn, proved!

4) Let $S = 1, 4, 7, 9$ and $T = 1, 2, 7, 8$. Explicitly list the members of the following sets: $S \cup T$, $S \cap T$, $S - T$, $S \times T$, $T \times S$, $\varphi(S)$ *and* $\varphi(T)$.

$S \cup T = \{1, 2, 4, 7, 8, 9\}$
$S \cap T = \{1, 7\}$
$S - T = \{4, 9\}$
$S \times T = \{(1,1),(1,2),(1,7),(1,8),(4,1),(4,2),(4,7),(4,8),(7,1),(7,2),(7,7),(7,8),(9,1),(9,2),(9,7),(9,8)\}$
$T \times S = \{(1,1),(1,4),(1,7),(1,9),(2,1),(2,4),(2,7),(2,9),(7,1),(7,4),(7,7),(7,9),(8,1),(8,4),(8,7),(8,9)\}$
$\varphi(S) = \{\emptyset, \{1\}, \{4\}, \{7\}, \{9\}, \{1,4\}, \{1,7\}, \{1,9\}, \{4,7\} \{4,9\}, \{7,9\}, \{1,4,7\}, \{1,4,9\}, \{4,7,9\}, \{1,4,7,9\}\}$
$\varphi(T) = \{\emptyset, \{1\}, \{2\}, \{7\}, \{8\}, \{1,2\}, \{1,7\}, \{1,8\}, \{2,7\} \{2,8\}, \{7,8\}, \{1,2,7\}, \{1,2,8\}, \{2,7,8\}, \{1,2,7,8\}\}$

5) Use set laws to prove that the two following sets are equivalent. $(1) A \cup B$ $(2)(A \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap B)$

$$
\begin{aligned}
(A \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap B) &= ((A \cap B) \cup (A \cap \overline{B})) \cup (\overline{A} \cap B) && \text{Associative Laws} \\
&= (A \cap (B \cup \overline{B})) \cup (\overline{A} \cap B) && \text{Distributive Laws} \\
&= (A \cap U) \cup (\overline{A} \cap B) && \text{Inverse Laws} \\
&= A \cup (\overline{A} \cap B) && \text{Identity Laws} \\
&= (A \cup \overline{A}) \cap (A \cup B) && \text{Distributive Laws} \\
&= U \cap (A \cup B) && \text{Inverse Laws} \\
&= A \cup B && \text{Identity Laws}
\end{aligned}
$$

6) Let A, B and C be arbitrary sets taken from the positive integers. Prove or disprove: If $A \cap B \cap C = \emptyset$, then $(A \subseteq \overline{B}) \vee (A \subseteq \overline{C})$

According to the question, we are given $\neg\{\exists x | x \in A \cap x \in B \cap x \in C\}$ or in English, there is NO same element in all A, B and C sets, to prove or disprove, either 1)if $x \in A$, then $x \in \overline{B}$ or 2)if $x \in A$, then $x \in \overline{C}$.

Disprove by counter example:
Let $A = \{1, 2\}, B = \{1\}, C = \{2\}, A \cap B \cap C = \emptyset$, $\overline{B} = \{X | \text{all positive integers except } 1\}$, $\overline{C} = \{X | \text{all positive integers except } 2\}$. Since $1 \in A$, and $1 \notin \overline{B}$, it follows that for this example, $(A \not\subseteq \overline{B})$. Since $2 \in A$, and $2 \notin \overline{C}$, it follows that for this example, $(A \not\subseteq \overline{C})$. This shows the statement isn't true for this example, so it can not be true for all sets A, B and C.

7) Give a summary of the life and mathematical contributions of Augustus De Morgan. Please aim for a length of roughly 200 - 400 words. Your summary must be typed. Please state the sources you used in writing your summary.

Augusta De Morgan (1806-1871) was born in India and his father was an Indian Army colonel. De Morgan emigrated to England at the age of seven months. He went to a private school where he developed an interest in mathematics. Demogan graduated from Trinity College in 1827. Although he thought of studying medicine or studying law, but he decided to take mathematics as his lifelong career in the end. He got a position at University College in London in 1828, but soon he got fired for no reason. However, after his successor's death in 1836, he returned to his position until 1866. De Morgan is famous for emphasizing that principle is better than technology. Many of his students are famous mathematicians, including Countess Augusta, a co-operative of the Babbage Computer research. De Morgan is a prolific writer who has authored more than 1000 articles for more than 15 journals. De Morgan also wrote textbooks for many disciplines, including logic, probabilities, calculus and algebra, and in 1838 he gave the first clear explanation of the important proof of mathematical induction, the term "mathematical induction", which he created and introduced. He made a groundbreaking contribution to the development of symbolic logic in the the 1840s. He invented the symbols that helped him to prove the equivalence of propositions, including the law named after him.